Dynamic Games with Complete Information

Lorenzo Rocco

Galilean School - University of Padova

April 2015

Dynamic games

- Why do we care of dynamic games? We already know that we can represent them in normal form and apply the concept of Nash Equilibrium.
- Unfortunately the Nash Equilibrium is too weak in the context of truly dynamic games: in these games it is possible that successive players try to influence preceding rivals by making empty threats, i.e. by announcing strategies that prescribe to play a suboptimal action at a given information set.
- We look for and apply refinements of the NE concept and for this we need to study the dynamic games in their extensive form.

Dynamic games with complete information

We focus on two classes of games:

- games with perfect information
- multi-stage game (possibly infinite multi-stage games)

where no player has private information

Definitions

Definition

Games of perfect information are games where at each stage only one player moves.

Definitions

Multi-stage games are composed of several stages, possibly infinite, where

1. all actions previously played in the game are known to all players
2. all players play simultaneously at each stage (and do not know what rivals are playing at that stage)

Sequential Rationality

To rule out non-credible threats we need:
Sequential rationality: *all players play their best response at each information set*.

- In games with perfect information it follows that players play their *best* action at each decision node.
- Hence, if we believe that *players' behavior satisfies sequential rationality, and that this is common knowledge,* then we can predict the best strategy that each player can play by solving the game starting from the end, from the next-to-last node (i.e. excluding terminal nodes) (**backward induction principle).**
- Therefore in games with perfect information each player can anticipate the actions that the following players play at their decision nodes.

Zermelo's Theorem

Theorem

Every finite game of perfect information has a NE in pure strategies that can be derived through backward induction. Moreover if no player has the same payoff at any two terminal nodes, then there is a unique Nash Equilibrium that satisfies the backward induction principle.

Note: how much rationality for backward induction? Think at the centipede game: player 1 and 2 start with 1 dollar each. They alternate declaring stop/continue. If a player says continue a referee takes 1 dollar from him and gives 2 dollars to the opponent. If a player says stop the game ends. Otherwise the game ends when both players have 100 dollars. Backward induction outcome is "say stop immediately". This is the worst outcome: but if player 2 thought that player 1 were not fully rational...

Exercise.

There are 2 firms $i = 1, 2$. Let q_i denote the quantity (integer) produced by firm $i = 1, 2$ at cost $4q_i$.Each firm cannot produce more than 3. The total demand is $P(Q) = 6 - Q$ where $Q = q_1 + q_2$. Firm 1 moves first choosing the quantity it wants to produce; firm 2 moves having observed the quantity produced by firm 1. Let describe the set of pure strategy of each player. Find a NE of the game in which firms produce the Cournot quantity. Find a NE in which firm 2 produced the monopoly outcome and firm 1 produces zero. Find the outcome ob backward induction of the game.

Subgames

The backward induction procedure can be generalized to all dynamic games and in particular to multi-stage games.
Preliminary we need to define:

Definition

A subgame of an extensive form game is a subset of the game having the following properties:

- it begins with an information set which is a singleton and contains all the decision nodes that are successors of this node, and only these nodes;
- if a decision node x is in the subgame then every $x' \in H(x)$ belongs to this subgame, where $H(x)$ is the information set containing the node x.

Subgames (cont.)

Note:

- the entire game is a subgame
- in games of perfect information every decision node initiates a subgame
- taken in isolation, a subgame is a game in its own right

Subgame Perfect Nash Equilibrium

Definition

A profile of strategy $\sigma = (\sigma_1, ..., \sigma_n)$ is a SPNE of a game if it induces a NE in every subgame.

Note:

- a SPNE is a NE because the entire game is itself a subgame
- in games of perfect information the set of SPNE coincides with the set of NE obtained by backward induction
- SPNE requires that players play NE strategies in each subgame, even in the subgame that would be played only as result of (other) players' mistake.

Finding SPNE

To identify the set of SPNE in general finite dynamic games, we can apply a generalized backward induction procedure:

1. start from the end and identify the final subgames (i.e. which do not contain a subgame in themselves)
2. select one NE in each of these final subgames and derive the reduced extensive form (i.e. substitute the NE payoff to the node where the subgame begins)
3. repeat steps 1 and 2.
4. if multiple equilibria are never encountered in any subgame, then there exists a unique SPNE.

Finding SPNE (cont.)

Theorem

In T-stage games, if there is a unique NE in each stage-game, then there is a unique SPNE which consists of each player playing the equilibrium strategy of the stage-game at each stage-game, independently of the history of the game

- Note: if multiple NE exists at each stage, then history dependent strategies can be played: a player might promise later rewards or punishments to influence current rivals' actions.
- Note: when games have infinite horizon we cannot use backward induction, but the concept of SPNE is still defined. However infinite horizon allows to play many credible strategies so that many SPNE will exist (eg. repeated Cournot game). Thus, typically SPNE loses its appeal

History dependent strategies

Recall: in multi-stage games,if only one NE exists in each stage-game, there will be a unique and history-independent SPNE

This is not the case when there are many NE in the stage game.

Example: consider this stage-game repeated twice, where payoff is the sum of stage payoffs

	L	M	R
U	0,-2	-1,-1	0,-2
M	4,3	-2,0	6,0
D	2,3	-2,0	5,5

- There are two stage-game NE: (M,L), (U,M) with payoffs (4,3) and (-1,-1)
- In the two-stage game the following strategy profile is a SPNE: "Play (D,R) in the first stage. If the first stage outcome is (D,R), then play (M,L) in the second stage. Otherwise play (U,M) in the second stage"

(cont.)

- In any second stage subgame, this strategy profile prescribes to play a NE
- In the sub-game starting in the first stage, i.e. the whole game, the payoff matrix resulting from the reduced extensive form obtained by applying generalized backward induction is

	L	M	R
U	0-1,-2-1	-1-1,-1-1	0-1,-2-1
M	4-1,3-1	-2-1,0-1	6-1,0-1
D	2-1,3-1	-2-1,0-1	5+4,5+3

- Now (D,R) is a Nash Equilibrium of the subgame
- Note: (D,R) is not an equilibrium of the stage-game

Payoffs in multi-stage games

In multi-stage games we assume that payoffs are:

- either the discounted sum of the stage-game payoffs

$$U_i = \sum_{t=0}^{T} \delta^t g_i(a_t) \quad \text{where} \quad a_t = (a_{t1} \ldots a_{tI})$$

- or the average discounted payoff

$$U_i = \frac{1-\delta}{1-\delta^{T-1}} \sum_{t=0}^{T} \delta^t g_i(a_t)$$

which gives the average stage-payoff independently of T. Example: if the stage-game payoffs are always v, the average discounted payoff is v for any T.

Strategies in multi-stage games

- Let the first period be labeled $t = 0$. The last period is T, so we have a total of $T + 1$ periods.
- The actions played in the stage game G are

$$a_t = (a_{t1} ... a_{tI})$$

- The history of the game up to period t is a description of all the actions played from the beginning of the game

$$h_t = (a_0, a_1, \ldots, a_{t-1})$$

- h_{T+1} is the history at the terminal nodes.
- H_t is the set of all possible histories up to period t

Strategies in multi-stage games (cont.)

To condition strategies on past events, they are made functions of history. So we write player i's period-t stage-game strategy as the function s_i^t where $s_i^t(h_t) = a_{ti}$ defined to all possible histories up to period t.

- Note: a player's stage-game action in any period and after any history must be drawn from her action space for that period.

Definition

A pure strategy for player i is a sequence of functions $\{s_i^t\}_{t=0}^T$ such that $s_i^t : H_t \to A_{ti}$ which specifies what a player must do at each stage for any previous history.

The strategy profile played at time t is

$$s^t = (s_1^t, ..., s_I^t)$$

Theorem

In every multi-stage game there exists an open-loop SPNE composed of history-independent strategies which induce a Nash equilibrium in each stage game.

Theorem

In every multi-stage game, if a closed-loop SPNE exists, i.e. such that strategies are history-dependent, the outcome induced in the last stage must be a Nash equilibrium of the last stage game.

Repeated games

- A particular stage-game is a repeated game, where the same stage is repeated each stage.
- In a repeated game, the actions available to each player in each stage are the same and the stage-payoffs depend only on the actions played in the stage-game.
- The environment for a repeated game is stationary (or, alternatively, independent of time and history). This *does not mean the actions themselves must be chosen independently of time or history*.

Example: repeated prisoner's dilemma.

Consider discounted payoff with common discount rate δ.

- finitely repeated: the only SPNE is (confess,confess)
- infinitely repeated: other outcomes are supported as SPNE although (confess, confess) is still a SPNE.

	Don′t confess	Confess
Don′t confess	-2,-2	-10,-1
Confess	-1,-10	-5,-5

trigger strategy: "not confess in the first period; continue in this way until no player deviates; after any deviation, confess for the rest of the game"

- Note: the trigger strategy induces an equilibrium in all subgames, especially those starting after a deviation, where (confess, confess) is a NE of the stage game.

(cont.)

Note:

- Patience is the key to support cooperation: in infinitely repeated games, even small future punishments can deter current deviation.
- In infinitely repeated games, the set of SPNE can be much different (and typically larger) than in finitely repeated games.

Folk theorem

Theorem

(Friedman, 1971) Given the infinitely repeated game Γ, let α^* be a stage-game equilibrium with payoffs e. Then for every payoff profile v with $v_i \geqslant e_i$ for all player i, there is a $\underline{\delta}$ such that for all $\delta > \underline{\delta}$ there is a SPNE of Γ with average (i.e. per period) payoffs v.

- Note: trigger strategies are played but here punishment consists in playing α^* after any deviation. Provided that individuals are patient enough this punishment is strong enough. Moreover, in each subgame these trigger strategies induce a NE.
- Example: infinitely repeated Cournot game. How many SPNE exists according to Friedman's Theorem?