whether this steady state exists is a matter of the order in which limits are taken. In any case, as noted in the text, the steady state at $k = 0$ has no economic content and is ignored throughout the book.

2.11 Exercises

2.1 Show that competitive labor markets and Assumption 1 imply that the wage rate must be strictly positive and thus (2.4) implies (2.3).

2.2 Prove that Assumption 1 implies that $F(A, K, L)$ is concave in K and L but not strictly so.

2.3 Show that when F exhibits constant returns to scale and factor markets are competitive, the maximization problem in (2.5) either has no solution (the firm can make infinite profits), a unique solution $K = L = 0$, or a continuum of solutions (i.e., any (K, L) with $K/L = \kappa$ for some $\kappa > 0$ is a solution).

2.4 Consider the Solow growth model in continuous time with the following per capita production function:

$$f(k) = k^4 - 6k^3 + 11k^2 - 6k.$$

(a) Which parts of Assumptions 1 and 2 does the underlying production function $F(K, L)$ violate?

(b) Show that with this production function, there exist three steady-state equilibria.

(c) Prove that two of these steady-state equilibria are locally stable, while one of them is locally unstable. Can any of these steady-state equilibria be globally stable?

2.5 Prove Proposition 2.7.

2.6 Prove Proposition 2.8.

2.7 Let us introduce government spending in the basic Solow model. Consider the basic model without technological change and suppose that (2.9) takes the form

$$Y(t) = C(t) + I(t) + G(t),$$

with $G(t)$ denoting government spending at time t. Imagine that government spending is given by $G(t) = \sigma Y(t)$.

(a) Discuss how the relationship between income and consumption should be changed. Is it reasonable to assume that $C(t) = sY(t)$?

(b) Suppose that government spending partly comes out of private consumption, so that $C(t) = (s - \lambda\sigma)Y(t)$, where $\lambda \in [0, 1]$. What is the effect of higher government spending (in the form of higher σ) on the equilibrium of the Solow model?

(c) Now suppose that a fraction ϕ of $G(t)$ is invested in the capital stock, so that total investment at time t is given by

$$I(t) = (1 - s - (1 - \lambda)\,\sigma + \phi\sigma)\,Y(t).$$

Show that if ϕ is sufficiently high, the steady-state level of capital-labor ratio will increase as a result of higher government spending (corresponding to higher σ). Is this reasonable? How would you alternatively introduce public investments in this model?

2.8 Suppose that $F(K, L, A)$ is concave in K and L (though not necessarily strictly so) and satisfies Assumption 2. Prove Propositions 2.2 and 2.5. How do we need to modify Proposition 2.6?

2.9 Prove Proposition 2.6.

2.10 Prove Corollary 2.2.

2.11 Consider a modified version of the continuous-time Solow growth model where the aggregate production function is

$$F(K, L, Z) = L^{\beta} K^{\alpha} Z^{1-\alpha-\beta},$$

where Z is land, available in fixed inelastic supply. Assume that $\alpha + \beta < 1$, capital depreciates at the rate δ, and there is an exogenous saving rate of s.

(a) First suppose that there is no population growth. Find the steady-state capital-labor ratio in the steady-state output level. Prove that the steady state is unique and globally stable.

(b) Now suppose that there is population growth at the rate n, that is, $\dot{L}/L = n$. What happens to the capital-labor ratio and output level as $t \to \infty$? What happens to returns to land and the wage rate as $t \to \infty$?

(c) Would you expect the population growth rate n or the saving rate s to change over time in this economy? If so, how?

2.12 Consider the continuous-time Solow model without technological progress and with constant rate of population growth equal to n. Suppose that the production function satisfies Assumptions 1 and 2. Assume that capital is owned by capitalists and labor is supplied by a different set of agents, the workers. Following a suggestion by Kaldor (1957), suppose that capitalists save a fraction s_K of their income, while workers consume all of their income.

(a) Define and characterize the steady-state equilibrium of this economy and study its stability.

(b) What is the relationship between the steady-state capital-labor ratio k^* and the golden rule capital stock k^*_{gold} defined in Section 2.2.3?

2.13 Let us now make the opposite assumption of Exercise 2.12 and suppose that there is a constant saving rate $s \in (0, 1)$ out of labor income and no savings out of capital income. Suppose that the aggregate production function satisfies Assumptions 1 and 2. Show that in this case multiple steady-state equilibria are possible.

* 2.14 In this exercise, you are asked to generalize Theorem 2.6 to a situation in which, rather than

$$\dot{Y}(t)/Y(t) = g_Y > 0,\ \dot{K}(t)/K(t) = g_K > 0, \text{ and } \dot{C}(t)/C(t) = g_C > 0$$

for all $t \geq T$ with $T < \infty$, we have

$$\dot{Y}(t)/Y(t) \to g_Y > 0,\ \dot{K}(t)/K(t) \to g_K > 0, \text{ and} \dot{C}(t)/C(t) \to g_C > 0.$$

(a) Show, by constructing a counterexample, that Part 1 of Theorem 2.6 is no longer correct without further conditions. [Hint: consider $g_c < g_k = g_y$.] What conditions need to be imposed to ensure that these limiting growth rates are equal to one another?

(b) Now suppose that Part 1 of Theorem 2.6 has been established (in particular, $g_Y = g_K$). Show that the equivalent of the steps in the proof of the theorem imply that for any T and $t \geq T$, we have

$$\exp\left(-\int_T^t g_Y(s)ds\right) Y(t)$$
$$= \tilde{F}\left[\exp\left(-\int_T^t g_K(s)ds\right) K(t),\ \exp\left(-n(t-T)\right) L(t),\ \tilde{A}(T)\right],$$

where $g_Y(t) \equiv \dot{Y}(t)/Y(t)$, and $g_K(t)$ and $g_C(t)$ are defined similarly. Then show that

$$Y(t) = \tilde{F}\left[\exp\left(\int_T^t \left(g_Y(s) - g_K(s)\right) ds\right) K(t), \exp\left(\int_T^t \left(g_Y(s) - n\right) ds\right) L(t), \tilde{A}(T)\right].$$

Next observe that for any $\varepsilon_T > 0$, there exists $T < \infty$, such that $\left|g_Y(t) - g_Y\right| < \varepsilon_T/2$ and $\left|g_K(t) - g_Y\right| < \varepsilon_T/2$ (from the hypotheses that $\dot{Y}(t)/Y(t) \to g_Y > 0$ and $\dot{K}(t)/K(t) \to g_K > 0$). Consider a sequence (or net; see Appendix A) $\left\{\varepsilon_T\right\} \to 0$, which naturally corresponds to $T \to \infty$ in the above definition. Take $t = \xi T$ for some $\xi > 1$, and show that Part 2 of Theorem 2.6 holds if $\varepsilon_T T \to 0$ (as $T \to \infty$). Using this argument, show that if both $g_Y(t)$ and $g_K(t)$ converge to g_Y and g_K at a rate strictly faster than $1/t$, the asymptotic production function has a representation of the form $F(K(t), A(t)L(t))$, but that this conclusion does not hold if either $g_Y(t)$ or $g_K(t)$ converges at a slower rate. [Hint: here an asymptotic representation means that $\lim_{t\to\infty} \tilde{F}/F = 1$.]

2.15 Recall the definition of the elasticity of substitution σ in (2.37). Suppose labor markets are competitive and the wage rate is equal to w. Prove that if the aggregate production function $F(K, L, A)$ exhibits constant returns to scale in K and L, then

$$\varepsilon_{y,w} \equiv \frac{\partial y/\partial w}{y/w} = \sigma,$$

where, as usual, $y \equiv F(K, L, A)/L$.

* 2.16 In this exercise you are asked to derive the CES production function (2.38) following the method in the original article by Arrow et al. (1961). These authors noted that a good empirical approximation to the relationship between income per capita and the wage rate was provided by an equation of the form

$$y = \alpha w^{\sigma},$$

where $y = f(k)$ is again output per capita and w is the wage rate. With competitive markets, recall that $w = f(k) - kf'(k)$. Thus the above equation can be written as

$$y = \alpha(y - ky')^{\sigma},$$

where $y = y(k) \equiv f(k)$ and y' denotes $f'(k)$. This is a nonlinear first-order differential equation.

(a) Using separation of variables (see Appendix B), show that the solution to this equation satisfies

$$y(k) = \left(\alpha^{-1/\sigma} + c_0 k^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}},$$

where c_0 is a constant of integration.

(b) Explain how you would put more structure on α and c_0 and derive the exact form of the CES production function in (2.38).

2.17 Consider the Solow growth model with constant saving rate s and depreciation rate of capital equal to δ. Assume that population is constant and the aggregate production function is given by the constant returns to scale production function

$$F(A_K(t)K(t), A_L(t)L(t)),$$

where $\dot{A}_L(t)/A_L(t) = g_L > 0$ and $\dot{A}_K(t)/A_K(t) = g_K > 0$.

(a) Suppose that F is Cobb-Douglas. Determine the BGP growth rate and the adjustment of the economy to the steady state.

(b) Suppose that F is not Cobb-Douglas (even asymptotically). Prove that there cannot exist $T < \infty$ such that the economy is on a BGP for all $t \geq T$. Explain why.

* 2.18 Consider the environment in Exercise 2.17. Suppose that F takes a CES form as in (2.38) with the elasticity of substitution between capital and labor $\sigma < 1$, $g_K > g_L$, and there is a constant saving rate s. Show that as $t \to \infty$, the economy converges to a BGP where the share of labor in national income is equal to 1 and capital, output, and consumption all grow at the rate g_L. In light of this result, discuss the claim in the literature that capital-augmenting technological change is inconsistent with balanced growth. Why is the claim in the literature incorrect? Relate your answer to Exercise 2.14.

* 2.19 In the context of Theorem 2.6, consider the production function

$$\tilde{F}(K(t), L(t), \tilde{A}(t)) = K(t)^{\tilde{A}(t)} L(t)^{1-\tilde{A}(t)},$$

where $\tilde{A}(t) : \mathbb{R}_+ \to (0, 1)$ is an arbitrary function of time, representing technology.

(a) Show that when $K(t) = \exp(nt)$ and $L(t) = \exp(nt)$ for $n \geq 0$, the conditions of Theorem 2.6 are satisfied and $\tilde{F}$ has a representation of the form $F(K(t), A(t)L(t))$. Determine a class of functions that provide such a representation.

(b) Show that the derivatives of $\tilde{F}$ and F are not equal.

(c) Suppose that factor markets are competitive. Show that while capital, output, and consumption grow at a constant rate, the capital share in national income behaves in an arbitrary fashion. [Hint: consider, for example, $\tilde{A}(t) = (2 + \sin(t))/4$.]

2.20 Consider the Solow model with noncompetitive labor markets. In particular, suppose that there is no population growth and no technological progress and output is given by $F(K, L)$. The saving rate is equal to s and the depreciation rate is given by δ.

(a) First suppose that there is a minimum wage $\bar{w}$, such that workers are not allowed to be paid less than $\bar{w}$. If labor demand at this wage falls short of L, employment is equal to the amount of labor demanded by firms, L^d (and the unemployed do not contribute to production and earn zero). Assume that $\bar{w} > f(k^*) - k^* f'(k^*)$, where k^* is the steady-state capital-labor ratio of the basic Solow model given by $f(k^*)/k^* = \delta/s$. Characterize the dynamic equilibrium path of this economy starting with some amount of physical capital $K(0) > 0$.

(b) Next consider a different form of labor market imperfection, whereby workers receive a fraction $\lambda > 0$ of output of their employer as their wage income. Characterize a dynamic equilibrium path in this case. [Hint: recall that the saving rate is still equal to s.]

2.21 Consider the discrete-time Solow growth model with constant population growth at the rate n, no technological change, and full depreciation (i.e., $\delta = 1$). Assume that the saving rate is a function of the capital-labor ratio and is thus given by $s(k)$.

(a) Suppose that $f(k) = Ak$ and $s(k) = s_0 k^{-1} - 1$. Show that if $A + \delta - n = 2$, then for any $k(0) \in (0, As_0/(1+n))$, the economy immediately settles into an asymptotic cycle and continuously fluctuates between $k(0)$ and $As_0/(1+n) - k(0)$. (Suppose that $k(0)$ and the parameters are given such that $s(k) \in (0, 1)$ for both $k = k(0)$ and $k = As_0/(1+n) - k(0)$.)

(b) Now consider the more general continuous production function $f(k)$ and saving function $s(k)$, such that there exist $k_1, k_2 \in \mathbb{R}_+$ with $k_1 \neq k_2$ and

$$k_2 = \frac{s(k_1) f(k_1) + (1-\delta) k_1}{1+n},$$

$$k_1 = \frac{s(k_2) f(k_2) + (1-\delta) k_2}{1+n}.$$

Show that when such (k_1, k_2) exist, there may also exist a stable steady state.

(c) Prove that such cycles are not possible in the continuous-time Solow growth model for any (possibly non-neoclassical) continuous production function $f(k)$ and continuous $s(k)$. [Hint: consider the equivalent of Figure 2.9.]

(d) What does the result in parts a–c imply for the approximations of discrete time by continuous time suggested in Section 2.4?

(e) In light of your answer to part d, what do you think of the cycles in parts a and b?

(f) Show that if $f(k)$ is nondecreasing in k and $s(k) = k$, cycles as in parts a and b are not possible in discrete time either.

2.22 Characterize the asymptotic equilibrium of the modified Solow/AK model mentioned in Section 2.6, with a constant saving rate s, depreciation rate δ, no population growth, and an aggregate production function of the form

$$F(K(t), L(t)) = A_K K(t) + A_L L(t).$$

2.23 Consider the basic Solow growth model with a constant saving rate s, constant population growth at the rate n, and no technological change, and suppose that the aggregate production function takes the CES form in (2.38).

(a) Suppose that $\sigma > 1$. Show that in this case equilibrium behavior can be similar to that in Exercise 2.22 with sustained growth in the long run. Interpret this result.

(b) Now suppose that $\sigma \to 0$, so that the production function becomes Leontief:

$$Y(t) = \min\left\{\gamma A_K(t)K(t); (1-\gamma)\ A_L(t)L(t)\right\}.$$

The model is then identical to the classical Harrod-Domar growth model developed by Roy Harrod and Evsey Domar (Harrod, 1939; Domar, 1946). Show that in this case there is typically no steady-state equilibrium with full employment and no idle capital. What happens to factor prices in these cases? Explain why this case is pathological, giving at least two reasons for why we may expect equilibria with idle capital or idle labor not to apply in practice.

2.24 Show that the CES production function (2.38) violates Assumption 2 unless $\sigma = 1$.

2.25 Prove Proposition 2.12.

2.26 Prove Proposition 2.13.

2.27 In this exercise, we work through an alternative conception of technology, which will be useful in the next chapter. Consider the basic Solow model in continuous time and suppose that $A(t) = A$, so that there is no technological progress of the usual kind. However, assume that the relationship between investment and capital accumulation is modified to

$$\dot{K}(t) = q(t)I(t) - \delta K(t),$$

where $\left[q(t)\right]_{t=0}^{\infty}$ is an exogenously given time-varying path (function). Intuitively, when $q(t)$ is high, the same investment expenditure translates into a greater increase in the capital stock.

Therefore we can think of $q(t)$ as the inverse of the relative price of machinery to output. When $q(t)$ is high, machinery is relatively cheaper. Gordon (1990) documented that the relative prices of durable machinery have been declining relative to output throughout the postwar era. This decline is quite plausible, especially given recent experience with the decline in the relative price of computer hardware and software. Thus we may want to suppose that $\dot{q}(t) > 0$. This exercise asks you to work through a model with this feature based on Greenwood, Hercowitz, and Krusell (1997).

(a) Suppose that $\dot{q}(t)/q(t) = \gamma_K > 0$. Show that for a general production function, $F(K, L)$, there exists no BGP.

(b) Now suppose that the production function is Cobb-Douglas, $F(K, L) = K^{\alpha} L^{1-\alpha}$, and characterize the unique BGP.

(c) Show that this steady-state equilibrium does not satisfy the Kaldor fact of constant K/Y. Is this discrepancy a problem? [Hint: how is K measured in practice? How is it measured in this model?]

8.12. References and Literature

The neoclassical growth model goes back to Frank Ramsey's (1928) classic treatment and for that reason is often referred to as the "Ramsey model". Ramsey's model is very similar to standard neoclassical growth model, except that it did not feature discounting. Another early optimal growth model was presented by John von Neumann (1935), focusing more on the limiting behavior of the dynamics in a linear model. The current version of the neoclassical growth model is most closely related to the analysis of optimal growth by David Cass (1965) and Tjalling Koopmans (1960, 1965). An excellent discussion of equilibrium and optimal growth is provided in Arrow and Kurz's (1970) volume.

All growth and macroeconomic textbooks cover the neoclassical growth model. Barro and Sala-i-Martin (2004, Chapter 2) provides a detailed treatment focusing on the continuous time economy. Blanchard and Fisher (1989, Chapter 2) and Romer (2006, Chapter 2) also present the continuous time version of the neoclassical growth model. Sargent and Ljungqvist (2004, Chapter 14) provides an introductory treatment of the neoclassical growth model in discrete time.

Ricardian Equivalence discussed in Exercise 8.19 was first proposed by Barro (1974). It is further discussed in Chapter 9.

A systematic quantitative evaluation of the effects of policy differences is provided in Chari, Kehoe and McGrattan (1997). These authors follow Jones (1995) in emphasizing differences in the relative prices of investment goods (compared to consumption goods) in the Penn Worlds tables and interpret these as due to taxes and other distortions. This interpretation is not without any problems. In particular, in the presence of international trade, these relative price differences will reflect other technological factors or possible factor proportion differences (see Chapter 19, and also Acemoglu and Ventura (2002) and Hsieh and Klenow (2006)). Parente and Prescott (1994) use an extended version of the neoclassical growth model (where the "stock of technology," which is costly to adopt from the world frontier, is interpreted as a capital good) to perform similar quantitative exercises. Other authors have introduced yet other accumulable factors in order to increase the elasticity of output to distortions (that is, to reduce the α parameter above). Pete Klenow has dubbed these various accumulable factors introduced in the models to increase this elasticity the "mystery capital" to emphasize the fact that while they may help the quantitative match of the neoclassical-type models, they are not directly observable in the data.

8.13. Exercises

Exercise 8.1. Consider the consumption allocation decision of an infinitely-lived household with (a continuum of) $L(t)$ members at time t, with $L(0) = 1$. Suppose that the household has total consumption $C(t)$ to allocate at time t. The household has "utilitarian" preferences with instantaneous utility function $u(c)$ and discount the future at the rate $\rho > 0$.

(1) Show that the problem of the household can be written as

$$\max \int_0^\infty \exp(-\rho t) \left[\int_0^{L(t)} u(c_i(t))\, di \right] dt,$$

subject to

$$\int_0^{L(t)} c_i(t)\, di \leq C(t),$$

and subject to the budget constraint of the household,

$$\dot{\mathcal{A}}(t) = r(t)\mathcal{A}(t) + W(t) - C(t),$$

where i denotes a generic member of the household, $\mathcal{A}(t)$ is the total asset holding of the household, $r(t)$ is the rate of return on assets and $W(t)$ is total labor income.

(2) Show that as long as $u(\cdot)$ is strictly concave, this problem becomes

$$\max \int_0^\infty \exp(-(\rho - n)t)\, u(c(t))\, dt,$$

subject to

$$\dot{a}(t) = (r(t) - n)\, a(t) + w(t) - c(t),$$

where $w(t) \equiv W(t)/L(t)$ and $a(t) \equiv \mathcal{A}(t)/L(t)$. Provide an intuition for this transformed problem.

Exercise 8.2. Derive (8.7) from (8.9).

Exercise 8.3. Suppose that the consumer problem is formulated without the no-Ponzi game condition. Construct a sequence of feasible consumption decisions that provides strictly greater utility than those characterized in the text.

Exercise 8.4. Consider a variant of the neoclassical model (with constant population growth at the rate n)in which preferences are given by

$$\max \int_0^\infty \exp(-\rho t)\, u(c(t))\, dt,$$

and there is population growth at the constant rate n. How does this affect the equilibrium? How does the transversality condition need to be modified? What is the relationship between the rate of population growth, n, and the steady-state capital labor ratio k^*?

Exercise 8.5. Prove Proposition 8.3.

Exercise 8.6. Explain why the steady state capital-labor ratio k^* does not depend on the form of the utility function without technological progress but depends on the intertemporal elasticity of substitution when there is positive technological progress.

Exercise 8.7. (1) Show that the steady-state saving rate s^* defined in (8.23) is decreasing in ρ, so that lower discount rates lead to higher steady-state savings.

(2) Show that in contrast to the Solow model, the saving rate s^* can never be so high that a decline in savings (or an increase in ρ) can raise the steady-state level of consumption per capita.

EXERCISE 8.8. In the dynamics of the basic neoclassical growth model, depicted in Figure 8.1, prove that the $\dot{c} = 0$ locus intersects the $\dot{k} = 0$ locus always to the left of k_{gold}. Based on this analysis, explain why the modified golden rule capital-labor ratio, k^*, given by (8.21) differs from k_{gold}.

EXERCISE 8.9. Prove that, as stated in Proposition 8.7, in the neoclassical model with labor-augmenting technological change and the standard assumptions, starting with $k(0) > 0$, there exists a unique equilibrium path where normalized consumption and capital-labor ratio monotonically converge to the balanced growth path. [Hint: use Figure 8.1].

EXERCISE 8.10. Consider a neoclassical economy, with a representative household with preferences at time $t = 0$:

$$U(0) = \int_0^\infty \exp(-\rho t) \frac{c(t)^{1-\theta} - 1}{1 - \theta}.$$

There is no population growth and labor is supplied inelastically. Assume that the aggregate production function is given by $Y(t) = F[A(t) K(t), L(t)]$ where F satisfies the standard assumptions (constant returns to scale, differentiability, Inada conditions).

(1) Define a competitive equilibrium for this economy.
(2) Suppose that $A(t) = A(0)$ and characterize the steady-state equilibrium. Explain why the steady-state capital-labor ratio is independent of θ.
(3) Now assume that $A(t) = \exp(gt) A(0)$, and show that a balanced growth path (with constant capital share in national income and constant and equal rates of growth of output, capital and consumption) exists only if F takes the Cobb-Douglas form, $Y(t) = (A(t) K(t))^\gamma (L(t))^{1-\gamma}$.
(4) Characterize the balanced growth path in the Cobb-Douglas case. Derive the common growth rate of output, capital and consumption. Explain why the (normalized) steady-state capital-labor ratio now depends on θ.

EXERCISE 8.11. Consider the baseline neoclassical model with no technological progress.

(1) Show that in the neighborhood of the steady state k^*, the law of motion of $k(t) \equiv K(t) / L(t)$ can be written as

$$\log[k(t)] = \log[k^*] + \eta_1 \exp(\xi_1 t) + \eta_2 \exp(\xi_2 t),$$

where ξ_1 and ξ_2 are the eigenvalues of the linearized system.
(2) Compute these eigenvalues show that one of them, say ξ_2, is positive.
(3) What does this imply about the value of η_2?
(4) How is the value of η_1 determined?

(5) What determines the speed of adjustment of $k(t)$ towards its steady-state value k^*?

EXERCISE 8.12. Derive closed-form equations for the solution to the differential equations of transitional dynamics presented in Example 8.2 with log preferences.

EXERCISE 8.13. (1) Analyze the comparative dynamics of the basic model in response to unanticipated increase in the rate of labor-augmenting technological progress will increase to $g' > g$. Does consumption increase or decrease upon impact?

(2) Analyze the comparative dynamics in response to the announcement at time T that at some future date $T' > T$ the discount rate will decline to $\rho' < \rho$. Does consumption increase or decrease at time T. Explain.

EXERCISE 8.14. Consider the basic neoclassical growth model with technological change and CRRA preferences (8.30). Explain why $\theta > 1$ ensures that the transversality condition is always satisfied.

EXERCISE 8.15. Consider a variant of the neoclassical economy with preferences given by

$$U(0) = \int_0^\infty \exp(-\rho t) \frac{(c(t) - \gamma)^{1-\theta} - 1}{1-\theta}$$

with $\gamma > 0$. There is no population growth. Assume that the production function is given by $Y(t) = F[K(t), A(t)L(t)]$, which satisfies all the standard assumptions and $A(t) = \exp(gt)A(0)$.

(1) Interpret the utility function.
(2) Define the competitive equilibrium for this economy.
(3) Characterize the equilibrium of this economy. Does a balanced growth path with positive growth in consumption exist? Why or why not?
(4) Derive a parameter restriction ensuring that the standard transversality condition is satisfied.
(5) Characterize the transitional dynamics of the economy.

EXERCISE 8.16. Consider a world consisting of a collection of closed neoclassical economies $\mathcal{J}$. Each $j \in \mathcal{J}$ has access to the same neoclassical production technology and admits a representative household with preferences $(1-\theta)^{-1}\int_0^\infty \exp(-\rho_j t)\left(c_j^{1-\theta} - 1\right)dt$. Characterize the cross-country differences in income per capita in this economy. What is the effect of the 10% difference in discount factor (e.g., a difference between a discount rate of 0.02 versus 0.022) on steady-state per capita income differences? [Hint: use the fact that the capital share of income is about 1/3].

EXERCISE 8.17. Consider the standard neoclassical growth model augmented with labor supply decisions. In particular, there is a total population normalized to 1, and all individuals have utility function

$$U(0) = \int_0^\infty \exp(-\rho t)\, u(c(t), 1 - l(t)),$$

where $l(t) \in (0,1)$ is labor supply. In a symmetric equilibrium, employment $L(t)$ is equal to $l(t)$. Assume that the production function is given by $Y(t) = F[K(t), A(t)L(t)]$, which satisfies all the standard assumptions and $A(t) = \exp(gt)A(0)$.

(1) Define a competitive equilibrium.
(2) Set up the current value Hamiltonian that each household solves taking wages and interest rates as given, and determine first-order conditions for the allocation of consumption over time and leisure-labor trade off.
(3) Set up the current value Hamiltonian for a planner maximizing the utility of the representative household, and derive the necessary conditions for an optimal solution.
(4) Show that the two problems are equivalent given competitive markets.
(5) Show that unless the utility function is asymptotically equal to

$$u(c(t), 1-l(t)) = \begin{cases} \frac{Ac(t)^{1-\theta}}{1-\theta} h(1-l(t)) & \text{for } \theta \neq 1, \\ A\log c(t) + Bh(1-l(t)) & \text{for } \theta = 1, \end{cases}$$

for some $h(\cdot)$ with $h'(\cdot) > 0$, there will not exist a balanced growth path with constant and equal rates of consumption and output growth, and a constant level of labor supply.

Exercise 8.18. Consider the standard neoclassical growth model with a representative household with preferences

$$U(0) = \int_0^\infty \exp(-\rho t)\left[\frac{c(t)^{1-\theta}-1}{1-\theta} + G(t)\right],$$

where $G(t)$ is a public good financed by government spending. Assume that the production function is given by $Y(t) = F[K(t), L(t)]$, which satisfies all the standard assumptions, and the budget set of the representative household is $C(t) + I(t) \leq Y(t)$, where $I(t)$ is private investment. Assume that $G(t)$ is financed by taxes on investment. In particular, the capital accumulation equation is

$$\dot{K}(t) = (1-\tau(t))I(t) - \delta K(t),$$

and the fraction $\tau(t)$ of the private investment $I(t)$ is used to finance the public good, i.e., $G(t) = \tau(t)I(t)$.

Take the sequence of tax rates $[\tau(t)]_{t=0}^{\infty}$ as given.

(1) Define a competitive equilibrium.
(2) Set up the individual maximization problem and characterize consumption and investment behavior.
(3) Assuming that $\lim_{t\to\infty}\tau(t) = \tau$, characterize the steady state.

(4) What value of τ maximizes the level of utility at the steady state. Starting away from the state state, is this also the tax rate that would maximize the initial utility level? Why or why not?

Exercise 8.19. Consider the neoclassical growth model with a government that needs to finance a flow expenditure of G. Suppose that government spending does not affect utility and that the government can finance this expenditure by using lump-sum taxes (that is, some amount $\mathcal{T}(t)$ imposed on each household at time t irrespective of their income level and capital holdings) and debt, so that the government budget constraint takes the form

$$\dot{b}(t) = r(t)\, b(t) + g - \mathcal{T}(t),$$

where $b(t)$ denotes its debt level. The no-Ponzi-game condition for the government is

$$\lim_{t\to\infty}\left[b(t)\exp\left(-\int_0^t (r(s)-n)\,ds\right)\right] = 0.$$

Prove the following *Ricardian equivalence* result: for any sequence of lump-sum taxes $[\mathcal{T}(t)]_{t=0}^{\infty}$ that satisfy the government's budget constraint (together with the no-Ponzi-game condition) leads of the same equilibrium sequence of capital-labor ratio and consumption. Interpret this result.

Exercise 8.20. Consider the baseline neoclassical growth model with no population growth and no technological change, and preferences given by the standard CRRA utility function (8.30). Assume, however, that the representative household can borrow and lend at the exogenously given international interest rate r^*. Characterize the steady state equilibrium and transitional dynamics in this economy. Show that if the economy starts with less capital than its steady state level it will immediately jump to the steady state level by borrowing internationally. How will the economy repay this debt?

Exercise 8.21. Modify the neoclassical economy (without technological change) by introducing cost of adjustment in investment as in the q-theory of investment studied in the previous chapter. Characterize the steady-state equilibrium and the transitional dynamics. What are the differences between the implications of this model and those of the baseline neoclassical model?

Exercise 8.22. * Consider a version of the neoclassical model that admits a representative household with preferences given by (8.30), no population growth and no technological progress. The main difference from the standard model is that there are multiple capital goods. In particular, suppose that the production function of the economy is given by

$$Y(t) = F(K_1(t), ..., K_M(t), L(t)),$$

where K_m denotes the m^{th} type of capital and L is labor. F is homogeneous of degree 1 in all of its variables. Capital in each sector accumulates in the standard fashion, with

$$\dot{K}_m(t) = I_m(t) - \delta_m K_m(t),$$

for $m = 1, ..., M$. The resource constraint of the economy is

$$C(t) + \sum_{m=1}^{M} I_m(t) \leq Y(t)$$

for all t.

(1) Write budget constraint of the representative household in this economy. Show that this can be done in two alternative and equivalent ways; first, with M separate assets, and second with only a single asset that is a claim to all of the capital in the economy.
(2) Define an equilibrium.
(3) Characterize the equilibrium by specifying the profit-maximizing decision of firms in each sector and the dynamic optimization problem of consumers.
(4) Write down the optimal growth problem in the form of a multi-dimensional current-value Hamiltonian and show that the optimum growth problem coincides with the equilibrium growth problem. Interpret this result.
(5) Characterize the transitional dynamics in this economy. Define and discuss the appropriate notion of saddle-path stability and show that the equilibrium is always saddle-path stable and the equilibrium dynamics can be reduced to those in the one-sector neoclassical growth model.
(6) Characterize the transitional dynamics under the additional assumption that investment is irreversible in each sector, i.e., $I_m(t) \geq 0$ for all t and each $m = 1, ..., M$.

EXERCISE 8.23. Contrast the effects of taxing capital income at the rate τ in the Solow growth model and the neoclassical growth model. Show that capital income taxes have no effect in the former, while they depress the effective capital-labor ratio in the latter. Explain why there is such a difference.

EXERCISE 8.24. Let us return to the discrete time version of the neoclassical growth model. Suppose that the economy admits a representative household with log preferences (i.e., $\theta = 1$ in terms of (8.30)) and the production function is Cobb-Douglas. Assume also that $\delta = 1$, so that there is full depreciation. Characterize the steady-state equilibrium and derive a difference equation that explicitly characterizes the behavior of capital stock away from the steady state.

Exercise 8.25. Again in the discrete time version of the neoclassical growth model, suppose that there is labor-augmenting technological progress at the rate g, i.e.,

$$A(t+1) = (1+g)\,A(t).$$

For simplicity, suppose that there is no population growth.

(1) Prove that balanced growth requires preferences to take the CRRA form
$$U(0) = \begin{cases} \sum_{t=0}^{\infty} \beta^t \frac{[c(t)]^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \sum_{t=0}^{\infty} \beta^t \ln c(t) & \text{if } \theta = 1 \end{cases}.$$
(2) Assuming this form of preferences, prove that there exists a unique steady-state equilibrium in which effective capital-labor ratio remains constant.
(3) Prove that this steady-state equilibrium is globally stable and convergence to this steady-state starting from a non-steady-state level of effective capital-labor ratio is monotonic.

while Barro (1974) is the reference for the original statement of the Ricardian Equivalence hypothesis. Another important application of overlapping generations models is to generational accounting, for example, as in the work by Auerbach and Kotlikoff (1987).

9.11. Exercises

Exercise 9.1. Prove that the allocation characterized in Proposition 9.1 is the unique competitive equilibrium. [Hint: first, show that there cannot be any equilibrium with $p_j > p_{j-1}$ for any j. Second, show that even if $p_0 > p_1$, household $i = 0$ must consume only commodity $j = 0$; then inductively, show that this is true for any household].

Exercise 9.2. Consider the following variant of economy with infinite number of commodities and infinite number of individuals presented in Section 9.1. The utility of individual indexed $i = j$ is

$$u\left(c\left(j\right)\right) + \beta u\left(c\left(j+1\right)\right)$$

where $\beta \in (0, 1)$, and each individual has one unit of the good with the same index as his own.

(1) Define a competitive equilibrium for this economy.
(2) Characterize the set of competitive equilibria in this economy.
(3) Characterize the set of Pareto optima in this economy.
(4) Can all Pareto optima be decentralized without changing endowments? Can they be decentralized by changing endowments?

Exercise 9.3. Show that in the model of Section 9.2 the dynamics of capital stock are identical to those derived in the text even when $\delta < 1$.

Exercise 9.4. In the baseline overlapping generations model, verify that savings $s\left(w, R\right)$, given by (9.6), are increasing in the first argument, w. Provide conditions on the utility function $u\left(\cdot\right)$ such that they are also increasing in the second argument, the interest rate R.

Exercise 9.5. Prove Proposition 9.4

Exercise 9.6. Consider the canonical overlapping generations model with log preferences

$$\log\left(c_1\left(t\right)\right) + \beta \log\left(c_2\left(t\right)\right)$$

for each household. Suppose that there is population growth at the rate n. Individuals work only when they are young, and supply one unit of labor inelastically. Production technology is given by

$$Y\left(t\right) = A\left(t\right) K\left(t\right)^{\alpha} L\left(t\right)^{1-\alpha},$$

where $A\left(t+1\right) = \left(1+g\right) A\left(t\right)$, with $A\left(0\right) > 0$ and $g > 0$.

(1) Define a competitive equilibrium and the steady-state equilibrium.
(2) Characterize the steady-state equilibrium and show that it is globally stable.
(3) What is the effect of an increase in g on the equilibrium?

(4) What is the effect of an increase in β on the equilibrium? Provide an intuition for this result.

Exercise 9.7. Consider the canonical model with log preferences, $\log(c_1(t)) + \beta \log(c_2(t))$, and the general neoclassical technology $F(K, L)$ satisfying Assumptions 1 and 2. Show that multiple steady-state equilibria are possible in this economy.

Exercise 9.8. Consider again the canonical overlapping generations model with log preferences and Cobb-Douglas production function.

(1) Define a competitive equilibrium.
(2) Characterize the competitive equilibrium and derive explicit conditions under which the steady-state equilibrium is dynamically inefficient.
(3) Using plausible numbers argue whether or not dynamic inefficiency can arise in "realistic" economies.
(4) Show that when there is dynamic inefficiency, it is possible to construct an unfunded Social Security system which creates a Pareto improvement relative to the competitive allocation.

Exercise 9.9. Consider again the canonical overlapping generations model with log preferences and Cobb-Douglas production function, but assume that individuals now work in both periods of their lives.

(1) Define a competitive equilibrium and the steady-state equilibrium.
(2) Characterize the steady-state equilibrium and the transitional dynamics in this economy.
(3) Can this economy generate overaccumulation?

Exercise 9.10. Prove Proposition 9.7.

Exercise 9.11. Consider the overlapping generations model with fully funded Social Security. Prove that even when the restriction $s(t) \geq 0$ for all t is imposed, no fully funded Social Security program can lead to a Pareto improvement.

Exercise 9.12. Consider an overlapping generations economy with a dynamically inefficient steady-state equilibrium. Show that the government can improve the allocation of resources by introducing national debt. [Hint: suppose that the government borrows from the current young and redistributes to the current old, paying back the current young the following period with another round of borrowing]. Contrast this result with the Ricardian equivalence result in Exercise 8.19 in Chapter 8.

Exercise 9.13. Prove Proposition 9.8.

Exercise 9.14. Consider the baseline overlapping generations model and suppose that the equilibrium is dynamically efficient, i.e., $r^* > n$. Show that any unfunded Social Security system will increase the welfare of the current old generation and reduce the welfare of some future generation.

Exercise 9.15. Derive equation (9.31).

Exercise 9.16. Consider the overlapping generations model with warm glow preferences in Section 9.6, and suppose that preferences are given by

$$c(t)^{\eta} b(t+1)^{1-\eta},$$

with $\eta \in (0,1)$, instead of equation (9.21). The production side is the same as in Section 9.6. Characterize the dynamic equilibrium of this economy.

Exercise 9.17. Consider the overlapping generations model with warm glow preferences in Section 9.6, and suppose that preferences are given by $u_1(c_i(t)) + u_2(b_i(t))$, where u_1 and u_2 are strictly increasing and concave functions. The production side is the same as in the text. Characterize a dynamic equilibrium of this economy.

Exercise 9.18. Characterize the aggregate equilibrium dynamics and the dynamics of wealth distribution in the overlapping generations model with warm glow preferences as in Section 9.6, when the per capita production function is given by the Cobb-Douglas form $f(k) = Ak^{\alpha}$. Show that away from the steady state, there can be periods during which wealth inequality increases. Explain why this may be the case.

Exercise 9.19. Generalize the results in Section 9.6 to an environment in which the preferences of an individual of generation t are given by

$$u(c(t)) + v(b(t)),$$

where $c(t)$ denotes own consumption, $b(t)$ is bequests, and $u(\cdot)$ and $v(\cdot)$ are strictly increasing, continuously differentiable and strictly concave utility functions. Determine conditions on $u(\cdot)$ and $v(\cdot)$ such that aggregate dynamics are globally stable. Provide conditions on $u(\cdot)$ and $v(\cdot)$ to ensure that asymptotically all individuals tend to the same wealth level.

Exercise 9.20. Show that the steady-state capital labor ratio in the overlapping generations model with impure altruism (of Section 9.6) can lead to overaccumulation, i.e., $k^* > k_{gold}$.

Exercise 9.21. Prove that given the perpetual youth assumption and population dynamics in equation (9.34), at time $t > 0$, there will be $n(1+n-\nu)^{t-s}(1-\nu)^{s-1}$ s-year-olds for any $s \in \{1, 2, ..., t-1\}$

Exercise 9.22. * Consider the discrete time perpetual youth model discussed in Section 9.7 and assume that preferences are logarithmic. Characterize the steady-state equilibrium and the equilibrium dynamics of the capital-labor ratio.

Exercise 9.23. Consider the continuous time perpetual youth model of Section 9.8.

(1) Show that given $L(0) = 1$, the initial size of a cohort born at the time $\tau \geq 0$ is $\exp((n-\nu)\tau)$.

(2) Show that the probability that an individual born at the time τ is alive at time $t \geq \tau$ is $\exp(-\nu(t-\tau))$.
(3) Derive equation (9.40).
(4) Show that this equation would not apply at any finite time if the economy starts at $t = 0$ with an arbitrary age distribution.

Exercise 9.24. Derive equation (9.46). [Hint: first integrate the flow budget constraint of the individual, (9.41) using the transversality condition (9.45) and then use the Euler equation (9.44)].

Exercise 9.25. Generalize the analysis of the continuous time perpetual youth model of Section 9.8 to an economy with labor-augmenting technological progress at the rate g. Prove that the steady-state equilibrium is unique and globally (saddle-path) stable. What is the impact of a higher rate of technological progress?

Exercise 9.26. Linearize the differential equations (9.43) and (9.49) around the steady state, (k^*, c^*), and show that the linearized system has one negative and one positive eigenvalue.

Exercise 9.27. Determine the effects of n and ν on the steady-state equilibrium (k^*, c^*) in the continuous time perpetual youth model of Section 9.8.

Exercise 9.28. (1) Derive equations (9.52) and (9.53).
(2) Show that for ζ sufficiently large, the steady-state equilibrium capital-labor ratio, k^*, can exceed k_{gold}, so that there is overaccumulation. Provide an intuition for this result.

Exercise 9.29. Consider the continuous time perpetual youth model with a constant flow of government spending G. Suppose that this does not affect consumer utility and that lump-sum taxes $[\mathcal{T}(t)]_{t=0}^{\infty}$ are allowed. Specify the government budget constraint as in Exercise 8.19 in Chapter 8. Prove that contrary to the Ricardian Equivalence result in Exercise 8.19, the sequence of taxes affects the equilibrium path of capital-labor ratio and consumption. Interpret this result and explain the difference between the overlapping generations model and the neoclassical growth model.

Exercise 9.30. * Consider the continuous time perpetual youth model with labor income declining at the rate $\zeta > 0$.

(1) Show that if $n = 0$, $k^* \leq k_{gold}$ for any $\zeta > 0$.
(2) Show that there exists $\zeta > 0$ sufficiently high such that if $n > 0$ and $\nu = 0$, $k^* > k_{gold}$.

Exercise 9.31. Consider an economy with aggregate production function

$$Y(t) = AK(t)^{1-\alpha} L(t)^{\alpha}.$$

All markets are competitive, the labor supply is normalized to 1, capital fully depreciates after use, and the government imposes a linear tax on capital income at the rate τ, and uses the proceeds for government consumption. Consider two specifications of preferences:

- All agents are infinitely lived, with preferences

$$\sum_{t=0}^{\infty} \beta^t \ln c(t)$$

- An overlapping generations model where agents work in the first period, and consume the capital income from their savings in the second period. The preferences of a generation born at time t, defined over consumption when young and old, are given by

$$\ln c^y(t) + \beta \ln c^0(t)$$

(1) Characterize the equilibria in these two economies, and show that in the first economy, taxation reduces output, while in the second, it does not.

(2) Interpret this result, and in the light of this result discuss the applicability of models which try to explain income differences across countries with differences in the rates of capital income taxation.

with exogenous savings), but dismissed it as uninteresting. A more complete treatment of sustained neoclassical economic growth is provided in Jones and Manuelli (1990), who show that even convex models (with production function is that satisfy Assumption 1, but naturally not Assumption 2) are consistent with sustained long-run growth. Exercise 11.4 is a version of the convex neoclassical endogenous growth model of Jones and Manuelli.

Barro and Sala-i-Martin (2004) discuss a variety of two-sector endogenous growth models with physical and human capital, similar to the model presented in Section 11.2, though the model presented here is much simpler than similar ones analyzed in the literature.

Romer (1986) is the seminal paper of the endogenous growth literature and the model presented in Section 11.4 is based on this paper. Frankel (1962) analyzed a similar growth economy, but with exogenous constant saving rate. The importance of Romer's paper stems not only from the model itself, but from two other features. The first is its emphasis on potential non-competitive elements in order to generate long-run economic growth (in this case knowledge spillovers). The second is its emphasis on the non-rival nature of knowledge and ideas. These issues will be discussed in greater detail in the next part of the book.

Another paper that has played a major role in the new growth literature is Lucas (1988), which constructs an endogenous growth model similar to that of Romer (1986), but with human capital accumulation and human capital externalities. Lucas' model is also similar to the earlier contribution by Uzawa (1964). Lucas's paper has played two major roles in the literature. First, it emphasized the empirical importance of sustained economic growth and thus was instrumental in generating interest in the newly emerging endogenous growth models. Second, it emphasized the importance of human capital and especially of human capital externalities. Since the role of human capital was discussed extensively in Chapter 10, which also showed that the evidence for human capital externalities is rather limited, we focused on the Romer model rather than the Lucas model. It turns out that Lucas model also generates transitional dynamics, which are slightly more difficult to characterize than the standard neoclassical transitional dynamics. A version of the Lucas model is discussed in Exercise 11.20.

11.7. Exercises

Exercise 11.1. Derive equation (11.14).

Exercise 11.2. Prove Proposition 11.2.

Exercise 11.3. Consider the following continuous time neoclassical growth model:

$$U(0) = \int_0^\infty \exp(-\rho t) \frac{(c(t))^{1-\theta} - 1}{1-\theta},$$

with aggregate production function

$$Y(t) = AK(t) + BL(t),$$

where $A, B > 0$.

(1) Define a competitive equilibrium for this economy.
(2) Set up the current-value Hamiltonian for an individual and characterize the necessary conditions for consumer maximization. Combine these with equilibrium factor market prices and derive the equilibrium path. Show that the equilibrium path displays non-trivial transitional dynamics.
(3) Determine the evolution of the labor share of national income over time.
(4) Analyze the impact of an unanticipated increase in B on the equilibrium path.
(5) Prove that the equilibrium is Pareto optimal.

Exercise 11.4. Consider the following continuous time neoclassical growth model:

$$U(0) = \int_0^\infty \exp(-\rho t) \frac{(c(t))^{1-\theta} - 1}{1-\theta},$$

with production function

$$Y(t) = A\left[L(t)^{\frac{\sigma-1}{\sigma}} + K(t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}.$$

(1) Define a competitive equilibrium for this economy.
(2) Set up the current-value Hamiltonian for an individual and characterize the necessary conditions for consumer maximization. Combine these with equilibrium factor market prices and derive the equilibrium path.
(3) Prove that the equilibrium is Pareto optimal in this case.
(4) Show that if $\sigma \leq 1$, sustained growth is not possible.
(5) Show that if A and σ are sufficiently high, this model generates asymptotically sustained growth due to capital accumulation. Interpret this result.
(6) Characterize the transitional dynamics of the equilibrium path.
(7) What is happening to the share of capital in national income? Is this plausible? How would you modify the model to make sure that the share of capital in national income remains constant?
(8) Now assume that returns from capital are taxed at the rate τ. Determine the asymptotic growth rate of consumption and output.

Exercise 11.5. Derive equations (11.19) and (11.20).

Exercise 11.6. Consider the neoclassical growth model with Cobb-Douglas technology $y(t) = Ak(t)^\alpha$ (expressed in per capita terms) and log preferences. Characterize the equilibrium path of this economy and show that as $\alpha \to 1$, equilibrium path approaches that of the baseline AK economy. Interpret this result.

Exercise 11.7. Consider the baseline AK model of Section 11.1 and suppose that two otherwise-identical countries have different taxes on the rate of return on capital. Consider the following calibration of the model where $A = 0.15$, $\delta = 0.05$, $\rho = 0.02$, and $\theta = 3$.

Suppose that the first country has a capital income tax rate of $\tau = 0.2$, while the second country has a tax rate of $\tau' = 0.4$. Suppose that the two countries start with the same level of income in 1900 and experience no change in technology or policies for the next 100 years. What will be the relative income gap between the two countries in the year 2000? Discuss this result and explain why you do (or do not) find the implications plausible.

Exercise 11.8. Prove that the necessary conditions for consumer optimization in Section 11.2 lead to the conditions enumerated in (11.25).

Exercise 11.9. Prove Proposition 11.3.

Exercise 11.10. Prove that the competitive equilibrium of the economy in Section 11.2, characterized in Proposition 11.3, is Pareto optimal and coincides with the solution to the optimal growth problem.

Exercise 11.11. Show that the rate of population growth has no effect on the equilibrium growth rate of the economies studied in Sections 11.1 and 11.2. Explain why this is. Do you find this to be a plausible prediction?

Exercise 11.12. * Show that in the model of Section 11.3, if the Cobb-Douglas assumption is relaxed, there will not exist a balanced growth path with a constant share of capital income in GDP.

Exercise 11.13. Consider the effect of an increase in α on the competitive equilibrium of the model in Section 11.3. Why does it increase the rate of capital accumulation in the economy?

Exercise 11.14. Consider a variant of the model studied in Section 11.3, where the technology in the consumption-good sector is still given by (11.27), while the technology in the investment-good sector is modified to

$$I(t) = A\left(K_I(t)\right)^{\beta}\left(L_I(t)\right)^{1-\beta},$$

where $\beta \in (\alpha, 1)$. The labor market clearing condition requires $L_C(t) + L_I(t) \leq L(t)$. The rest of the environment is unchanged.

(1) Define a competitive equilibrium.
(2) Characterize the steady-state equilibrium and show that it does not involve sustained growth.
(3) Explain why the long-run growth implications of this model differ from those of Section 11.3.
(4) Analyze the steady-state income differences between two economies taxing capital at the rates τ and τ'. What are the roles of the parameters α and β in determining these relative differences? Why do the implied magnitudes differ from those in the one-sector neoclassical growth model?

Exercise 11.15. In the Romer model presented in Section 11.4, let g_C^* be the growth rate of consumption and g^* the growth rate of aggregate output. Show that $g_C^* > g^*$ is not feasible, while $g_C^* < g^*$ would violate the transversality condition.

Exercise 11.16. Consider the Romer model presented in Section 11.4. Prove that the allocation in Proposition 11.5 satisfies the transversality condition. Prove also that there are no transitional dynamics in this equilibrium.

Exercise 11.17. Consider the Romer model presented in Section 11.4 and suppose that population grows at the exponential rate n. Characterize the labor market clearing conditions. Formulate the dynamic optimization problem of a representative household and show that any interior solution to this problem violates the transversality condition. Interpret this result.

Exercise 11.18. Consider the Romer model presented in Section 11.4. Provide two different types of tax/subsidy policies that would make the equilibrium allocation identical to the Pareto optimal allocation.

Exercise 11.19. Consider the following infinite-horizon economy in discrete time that admits a representative household with preferences at time $t = 0$ as

$$U(0) = \sum_{t=0}^{\infty} \beta^t \left[\frac{C(t)^{1-\theta} - 1}{1-\theta} \right],$$

where $C(t)$ is consumption, and $\beta \in (0,1)$. Total population is equal to L and there is no population growth and labor is supplied inelastically. The production side of the economy consists of a continuum 1 of firms, each with production function

$$Y_i(t) = F(K_i(t), A(t) L_i(t)),$$

where $L_i(t)$ is employment of firm i at time t, $K_i(t)$ is capital used by firm i at time t, and $A(t)$ is a common technology term. Market clearing implies that $\int_0^1 K_i(t)\,di = K(t)$, where $K(t)$ is the total capital stock at time t,and $\int_0^1 L_i(t)\,di = L(t)$. Assume that capital fully depreciates, so that the resource constraint of the economy is

$$K(t+1) = \int_0^1 Y_i(t)\,di - C(t).$$

Assume also that labor-augmenting productivity at time t, $A(t)$, is given by

$$A(t) = K(t). \tag{11.42}$$

(1) Explain (11.42) and why it implies a (non-pecuniary) externality.
(2) Define a competitive equilibrium (where all agents are price takers—but naturally not all markets are complete).
(3) Show that there exists a unique balanced growth path competitive equilibrium, where the economy grows (or shrinks) at a constant rate every period. Provide a

condition on F, β and θ such that this growth rate is positive, but the transversality condition is still satisfied.

(4) Argue (without providing the math) why any equilibrium must be along the balanced growth path characterized in part 3 at all points.

(5) Is this a good model of endogenous growth? If yes, explain why. If not, contrast it with what you consider to be better models.

EXERCISE 11.20. * Consider the following endogenous growth model due to Uzawa and Lucas. The economy admits a representative household and preferences are given by

$$\int_0^\infty \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1 - \theta} dt,$$

where $C(t)$ is consumption of the final good, which is produced as

$$Y(t) = AK(t)^\alpha H_P^{1-\alpha}(t)$$

where $K(t)$ is capital and $H(t)$ is human capital, and $H_P(t)$ denotes human capital used in production. The accumulation equations are as follows:

$$\dot{K}(t) = I(t) - \delta K(t)$$

for capital and

$$\dot{H}(t) = BH_E(t) - \delta H(t)$$

where $H_E(t)$ is human capital devoted to education (further human capital accumulation), and the depreciation of human capital is assumed to be at the same rate as physical capital for simplicity (δ). The resource constraints of the economy are

$$I(t) + C(t) \leq Y(t)$$

and

$$H_E(t) + H_P(t) \leq H(t).$$

(1) Interpret the second resource constraint.

(2) Denote the fraction of human capital allocated to production by $\phi(t)$, and calculate the growth rate of final output as a function of $\phi(t)$ and the growth rates of accumulable factors.

(3) Assume that $\phi(t)$ is constant, and characterize the balanced growth path of the economy (with constant interest rate and constant rate of growth for capital and output). Show that in this balanced growth path, we have $r^* \equiv B - \delta$ and the growth rate of consumption, capital, human capital and output are given by $g^* \equiv (B - \delta - \rho)/\theta$. Show also that there exists a unique value of $k^* \equiv K/H$ consistent with balanced growth path.

(4) Determine the parameter restrictions to make sure that the transversality condition is satisfied.

(5) Now analyze the transitional dynamics of the economy starting with K/H different from k^* [Hint: look at dynamics in three variables, $k \equiv K/H$, $\chi \equiv C/K$ and ϕ, and consider the cases $\alpha < \theta$ and $\alpha \geq \theta$ separately].