The Solow Model

The basic model of exogneous growth

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Motivation to growth theory.

We talk of economic growth when *per capita* income increases.

- For the last 200 years western countries have experienced sustained growth
- In the second post-war period, Japan and Asian Tigers have experienced sustained growth
- Since the Eighties (?) China and India are experiencing sustained growth
- Since the Nineties Brazil and other Latin American countries are experiencing sustained growth
- At the same time African and Central Asian Countries languished and remained behind

Is it obvious that *per capita* income grow?
What are the (proximate) causes of growth?
Is it possible to trigger growth?
Why do only some countries grow?

Solow model in pills.

- According to Solow, growth depends on factors accumulation, especially capital accumulation.
- The economy accumulates capital by saving a part of income each period.
- Saving rate is exogenously fixed.

- However growth is a transitory phenomenon: there is no growth in the long run, if factor accumulation occurs alone.
- A sustained growth can be observed only if some technological progress occurs.

Hypothesis 1.

- Infinite horizon / continuous time
- closed economy
- representative household
- representative competitive firm
- households save the fraction s of their income each period
- savings $\Longrightarrow$ investment, i.e. new capital
- capital depreciates at rate δ (wear and tear of machinery)
- capital belongs to household
- household supply 1 unit of labor inelastically
- population grows at constant rate n

Hypothesis 2 - Technology

Representative firm's production function is

$$Y_t = F(K_t, L_t, A_t)$$

where

- Y_t is total output at period t
- K_t is capital stock at period t
- L_t is labor stock at period t
- A_t is technology at period t

- $F(.)$ is twice continuously differentiable in K_t and L_t
- $F_K > 0$ $\quad F_{KK} < 0$ diminishing returns to capital
- $F_L > 0$ $\quad F_{LL} < 0$ diminishing returns to labor
- constant returns to scale in K_t and L_t, i.e. $F(.)$ is homogenous of degree 1

Useful properties.

Definition

A function $F(K_t, L_t, A_t)$ is homogenous of degree one in K_t and L_t if

$$F(\lambda K_t, \lambda L_t, A_t) = \lambda F(K_t, L_t, A_t) \text{ for all } \lambda \in \mathbb{R}_+$$

Theorem

If $F(K_t, L_t, A_t)$ is homogenous of degree one in K_t and L_t, then

$$F(K_t, L_t, A_t) = F_K(.)K_t + F_L(.)L_t$$

Theorem

If $F(K_t, L_t, A_t)$ is homogenous of degree one in K_t and L_t, then

$$F_K(.) \text{ and } F_L(.) \text{ are homogenous of degree zero}$$

Hypothesis 3 - Technology

Inada conditions:

$$\lim_{K \to 0} F_K(.) = +\infty \qquad \lim_{K \to +\infty} F_K(.) = 0$$

$$\lim_{L \to 0} F_L(.) = +\infty \qquad \lim_{L \to +\infty} F_K(.) = 0$$

Firms.

Firms maximize profits in a competitive setting (price takers):

$$\max_{L_t, K_t} F(K_t, L_t, A_t) - w_t L_t - R_t K_t$$

(note: output price is normalized to 1 (numeraire)
FOCs are

$$F_L(.) = w_t$$

$$F_K(.) = R_t$$

Therefore:

- $\pi_t = F(K_t, L_t, A_t) - F_L(.)L_t - F_K(.)K_t = 0$ since $F(.)$ has constant returns to scale.
- The entire production goes to households

Capital accumulation.

- Households save the constant fraction s of their income, i.e. $S_t = sF(.)$
- In a closed economy, $S_t = I_t$

The cornerstone of the Solow model is the law of capital accumulation

$$\dot{K}_t = sF(K_t, L_t, A_t) - \delta K_t \tag{1}$$

where:

- $\dot{K}_t$ is the "instantaneous change" in the stock of capital from period t to period $t + \varepsilon$ (mathematically it is $\frac{\partial K_t}{\partial t}$)
- $sF(K_t, L_t, A_t)$ is investment in new capital at time t
- δK_t is capital depreciation at time t (at the end of period t only $(1-\delta)K_t$ is viable for future production)

In per-worker (=per-capita) units (1).

Define capital units per worker (capital-labor ratio) as

$$k_t = \frac{K_t}{L_t}$$

Note that

$$\dot{k}_t = \frac{\dot{K}_t L_t - \dot{L}_t K_t}{L_t^2} = \frac{\dot{K}_t}{L_t} - \frac{\dot{L}_t}{L_t} k_t = \frac{\dot{K}_t}{L_t} - n k_t$$

Recall: Population grows at constant rate n, i.e. $L_t = L_0 \exp(nt)$.
Therefore $\frac{\dot{L}_t}{L_t}$ is the population instantaneous growth rate:

$$\frac{\partial L_t}{\partial t} = n L_0 \exp(nt) \quad \text{so that} \quad \frac{\dot{L}_t}{L_t} = n$$

In per-worker (=per-capita) units (2).

Write (1) in per-capita terms

$$\frac{\dot{K}_t}{L_t} = s\frac{F(K_t, L_t, A_t)}{L_t} - \delta\frac{K_t}{L_t} \tag{2}$$

Note that, thanks to homogeneity,

$$\frac{1}{L_t}F(K_t, L_t, A_t) = F(\frac{1}{L_t}K_t, \frac{1}{L_t}L_t, A_t) = F(\frac{K_t}{L_t}, 1, A_t)$$

Define

$$f(k_t) = F(\frac{K_t}{L_t}, 1, A_t)$$

Therefore (2) can be written as

$$\dot{k}_t = sf(k_t) - (\delta + n)k_t \tag{3}$$

This is the fundamental equation of the Solow model

Equilibrium growth path

Definition

An equilibrium growth path for this model is a sequence

$$[K_t, L_t, Y_t, c_t, w_t, R_t]_{t=0}^{\infty}$$

such that

- K_t satisfies $\dot{k}_t = sf(k_t) - (\delta + n)k_t$
- $L_t = L_0 \exp(nt)$
- $Y_t = F(K_t, L_t, A_t)$
- $c_t = (1 - s)f(k_t)$
- $w_t = F_L(.)$
- $R_t = F_K(.)$

Steady state.

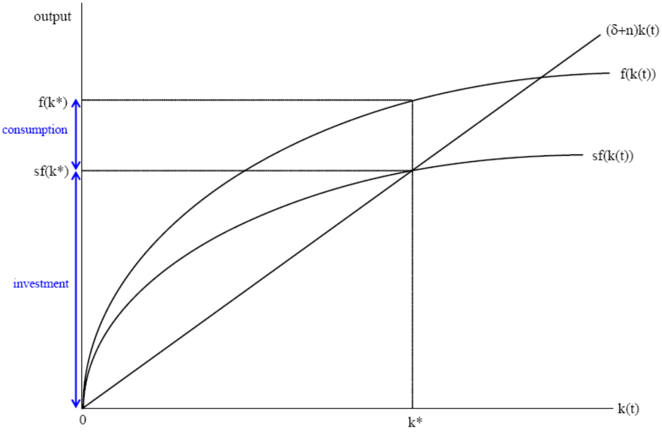
Definition

The steady state of this economy is fully characterized by the condition $\dot{k}_t = 0$, i.e. per capita units of capital remain constant when the economy is at a steady state.

Given $\dot{k}_t = sf(k_t) - (\delta + n)k_t$, the steady state capital-labor ratio is such that

$$\frac{f(k^*)}{k^*} = \frac{\delta + n}{s}$$

Steady state. Graphical representation



Note: $f(k^*)$ is concave because $F(.)$ is concave in K_t. Moreover $f_k(0) = \infty$ and $f_k(\infty) = 0$

Comparative statics at the steady state

The steady state level of k_t is a function of the economy parameters (exogenous variables)

$$k^* = k^*(n, \delta, s)$$

By looking at the diagram, we can check that

$$\frac{\partial k^*}{\partial n} < 0 \qquad \frac{\partial k^*}{\partial \delta} < 0 \qquad \frac{\partial k^*}{\partial s} > 0$$

- A higher population growth rate reduces steady state capital per worker $\Longrightarrow y^*(n) \downarrow$
- A higher depreciation rate reduces steady state capital per worker $\Longrightarrow y^*(n) \downarrow$
- A higher saving rate increases steady state capital per worker $\Longrightarrow y^*(n) \uparrow$

Stability 1

Is the steady state the "final outcome" if this economy? Or is the economy converging to the steady state whatever its initial condition?

Theorem

Suppose that $g(.)$ is continuos and there exists a unique x^* such that $g(x^*) = 0$. Moreover suppose that $g(x) < 0$ for $x > x^*$ and $g(x) > 0$ for $x < x^*$. Then the nonlinear differential equation $\dot{x}_t = g(x_t)$ is globally asymptotically stable, i.e. starting from any x_0, $x_t \rightarrow x^*$.

Let $g(k_t) = sf(k_t) - (\delta + n)k_t$. We know that:

- $g(k^*) = 0$ and k^* is the unique root of $g(.)$
- $g(k^*) > 0$ for $k_t < k^*$ (at low capital-per-worker levels, savings exceed "depreciation") –> capital accumulation
- $g(k^*) < 0$ for $k_t > k^*$ (at high capital-per-worker levels, savings fall short of "depreciation") –> capital de-cumulation

Stability 2

- The steady state of the Solow model is therefore globally stable. From whatever initial capital stock one economy starts, it converges to the steady state.
- *Note: at the steady state there is no growth of per-capita capital and per-capita income*
- This means that the current setting is unable to produce (and so explain) growth in the long run.
- Note: according to the Solow model, if any two countries shares the same technology and the same parameters, they will converge to the same steady state and at the steady state there will be no per-capita growth (Unconditional convergence).

Problem 1

Problem

Let $F(K_t, L_t, A_t) = AK_t^{\alpha}L_t^{1-\alpha}$ *with* $0 < \alpha < 1$. *Derive the steady state.*

Solution

$f(k_t) = \frac{1}{L_t}F(K_t, L_t, A_t) = F(\frac{K_t}{L_t}, 1, A_t) = A\left(\frac{K_t}{L_t}\right)^{\alpha} = Ak_t^{\alpha}$. *Therefore* $\dot{k}_t = 0$ *implies*

$$sAk_t^{\alpha} - (\delta + n)k_t = 0$$

whose solution is

$$k^* = \left(\frac{As}{\delta + n}\right)^{\frac{1}{1-\alpha}}$$

The corresponding steady-state per-capita income is

$$y^* = A\left(\frac{As}{\delta + n}\right)^{\frac{\alpha}{1-\alpha}}$$

Problem 2

Problem

Let $F(K_t, L_t, A_t) = AK_t^{\alpha} L_t^{1-\alpha}$ *with* $0 < \alpha < 1$. *Derive the transitional dynamics to the steady state.*

Solution

We know that the law of motion of the economy is $\dot{k}_t = sAk_t^{\alpha} - (\delta + n)k_t$ *with initial per-worker capital* k_0. *This law describes how the state variable* k_t *changes each instant. Therefore we simply need to integrate this equation to get the path of* k_t *from* k_0 *to* k^*.

- *The task is difficult because the law of motion is a non-linear differential equation. Substitute* $x_t = k_t^{1-\alpha}$
- *Then* $\dot{x}_t = (1-\alpha)k_t^{-a}\dot{k}_t = (1-\alpha)\frac{x_t}{k_t}\dot{k}_t$
- $\dot{k}_t = sAk_t^{\alpha} - (\delta + n)k_t \quad \Longleftrightarrow$
  $\dot{x}_t = (1-\alpha)\frac{x_t}{k_t}\left[sAk_t^{\alpha} - (\delta + n)k_t\right] = (1-\alpha)sA - (1-\alpha)(\delta + n)x_t$
  which is a linear differential equation in x_t

Solution (cont.)

The general solution for $\dot{x}_t = Ax_t + B$ *is* $x_t = -\frac{B}{A} + c\exp(At)$ *and the particular solution if the initial condition is* x_0 *is*
$x_t = -\frac{B}{A} + \left(x_0 - \frac{B}{A}\right)\exp(At)$

- *Therefore* $x_t = \frac{sA}{\delta+n} + \left(x_0 - \frac{sA}{\delta+n}\right)\exp(-(1-\alpha)(\delta+n)t)$
- *Substituting, we get*

$$k_t = \left[\frac{sA}{\delta+n} + \left(k_0^{1-\alpha} - \frac{sA}{\delta+n}\right)\exp(-(1-\alpha)(\delta+n)t)\right]^{\frac{1}{1-\alpha}}$$

Note that:

- for $t \rightarrow \infty$, $k_t \rightarrow \left[\frac{sA}{\delta+n}\right]^{\frac{1}{1-\alpha}}$, the steady state already computed in Problem 1
- The rate of convergence (i.e. the speed) is related to $(1-\alpha)(\delta+n)$

On the convergence

From the example above we observe:

- a higher α slows down convergence: the less diminishing are capital returns, the closer you are to the steady state and the slower you converge. Slow convergence in a sense means sustained growth.
- a higher δ (or n) accelerates convergence: it hampers capital accumulation and so the economy takes less time to get a smaller steady state.

Technological progress

The model implies no sustained growth (i.e. no growth in the long run). This is at odds with data: for instance western countries have grown for the last 200 years! How can we obtain sustained growth?

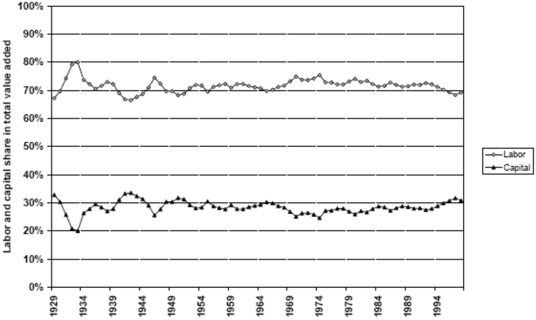
One candidate is the technological progress. So far we have assumed that $A_t = A$

Now we assume instead that productive efficiency increases over time (i.e. the same input mix will produce more) thanks to technological progress.

Yet, we need a form of technological progress which produces a balanced growth path (BGP).

Balanced growth 1

Balanced growth is something we observe in data (Kaldor facts):



Capital and Labor Share in the U.S. GDP.

Income shares of capital and labor remain constant if the stocks of income and capital growth at the same rate.

Technological Progress 2

There are three possible ways to model technological progress:

- Neutral technological progress (or Hicks-neutral):

$$F(K_t, L_t, A_t) = A_t \widetilde{F}(K_t, L_t)$$

- Capital augmenting tech. progress (or Solow-neutral):

$$F(K_t, L_t, A_t) = \widetilde{F}(A_t K_t, L_t)$$

- Labor augmenting tech. progress (or Harrod-neutral):

$$F(K_t, L_t, A_t) = \widetilde{F}(K_t, A_t L_t)$$

Uzawa Theorem

In order to get balanced growth we are forced to use only labor augmenting tech. progress (Uzawa theorem, 1961). In this case the growth rate of per-capita income and capital at steady state must be equal to the rate of technological progress

$$\frac{\dot{y}}{y} = \frac{\dot{k}}{k} = \frac{\dot{A}}{A} = g$$

The Solow model with technological progress 1

Let's assume

$$Y_t = F(K_t, A_t L_t)$$

where $F(.)$ has the usual properties. Assume that $A_t = A_0 \exp(gt)$.
We refer to the quantity $A_t L_t$ as effective units of labor.
The law of capital accumulation is again

$$\dot{K}_t = sF(K_t, A_t L_t) - \delta K_t \tag{4}$$

Define now

$$k_t = \frac{K_t}{A_t L_t}$$

The Solow model with technological progress 2

Some technicalities:

$$\dot{k}_t = \frac{\dot{K}_t A_t L_t - \left(\dot{A}_t L_t + A_t \dot{L}_t\right)}{\left(A_t L_t\right)^2} \text{ so that } \frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - g - n \qquad (5)$$

By substituting $\dot{K}_t$ in (5) we get

$$\frac{\dot{k}_t}{k_t} = \frac{sF(K_t, A_t L_t) - \delta K_t}{K_t} - g - n = \frac{sf(k_t)}{k_t} - (g + \delta + n) \qquad (6)$$

The Solow model with technological progress 3

The law of motion (6) is looks very much like the law of motion of the basic Solow model.
Therefore there is a unique and globally stable steady state as well in this augmented Solow model (i.e. $\dot{k}_t = 0$).
The steady state capital per unit of effective labor is such that

$$\frac{sf(k^*)}{k^*} = (g + \delta + n)$$

At the steady state, capital and output per unit of effective labor do not grow. However capital and output per capita do grow at the rate g.

The Solow model with technological progress 4

Define output per unit of effective labor as

$$\widehat{y}_t = \frac{Y_t}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, 1\right) = f(k_t)$$

and income per capita as

$$y_t = \frac{Y_t}{L_t} = A_t \widehat{y}_t$$

By differentiating wrt time we have

$$\dot{y}_t = \dot{A}_t \widehat{y}_t + \dot{\widehat{y}}_t A_t = \dot{A}_t \widehat{y}_t$$

and dividing by y_t

$$\frac{\dot{y}_t}{y_t} = \frac{\dot{A}_t \widehat{y}_t}{A_t \widehat{y}_t} = \frac{\dot{A}_t}{A_t} = g$$

The Solow model with technological progress 4

Now, the model generates balanced growth in the long run and fits empirical evidence much better. However such growth is completely exogenously driven. It entirely depends on technological progress, but where the technological progress come from?
The endogenous growth model that we will see provide an answer to this question.

The golden rule

Consider consumption: after all this is the variable we are ultimately interested in.
Per capita consumption is

$$c = y - sy$$

At the steady state we have

$$c^* = A_t f(k^*(s)) - sA_t f(k^*(s)) = A_t f(k^*(s)) - A_t(g + \delta + n)k^*(s)$$

It is easy to check that c^* is concave in s so that there exists a unique value of s which maximizes per-capita consumption

$$\frac{\partial c^*}{\partial s} = A_t \left(\frac{\partial f(k^*)}{\partial k^*} - (g + \delta + n) \right) \frac{\partial k^*}{\partial s}$$

which is zero if

$$\frac{\partial f(k^*)}{\partial k^*} = (g + \delta + n)$$

The golden rule 2

We define s_{gold} i.e. the saving rate of golden rule, that value of s which makes $\frac{\partial f(k^*)}{\partial k^*} = (g + \delta + n)$ and maximizes consumption per capita. The corresponding per capita capital of golden rule is denoted k_{gold}. At the golden rule the marginal productivity of capital is just enough to replenish the capital stock after depreciation and dilution due to population growth and technological progress (replacement rate).

If $s > s_{gold}$ there is too much capital accumulation and too little consumption, even if the economy is at equilibrium. This is a situation called "dynamic inefficiency", that we will encounter again in this course.

We want to decompose economic growth into its elements
Let $Y_t = F(K_t, L_t, A_t)$. Differentiating wrt to time gives

$$\dot{Y}_t = F_K \dot{K}_t + F_L \dot{L}_t + F_A \dot{A}_t$$

and dividing by Y_t gives

$$g = \alpha_K g_K + \alpha_L g_L + x$$

where α_i are the income shares to capital and labor, g_i are growth rates of capital and labor and x is the so called Solow residual or total factor productivity (TFP).
Therefore income growth can be decomposed in factors growth and TFP.
According to Solow (1957) a large part of US growth is due to TFP. So it is important to well understand what determines x.

The Solow Model and the Data 1.

The Solow model augmented with technological progress has been empirically tested in many papers, especially in its implication about convergence.
We want to explicitate the relationship between growth and distance from the steady state: intuitively countries far away from the steady state must growth faster (catching up) - look at the diagram of the Solow model.
From

$$y_t = A_t f(k_t)$$

differentiation wrt time and division by y_t gives

$$\frac{\dot{y}_t}{y_t} = \frac{\dot{A}_t}{A_t} + \frac{f_k(k_t)\dot{k}_t A_t}{A_t f(k_t)}$$

or in terms of elasticity of $f(k)$

$$\frac{\dot{y}_t}{y_t} = g + \varepsilon_k(k_t)\frac{\dot{k}_t}{k_t} \quad \text{where} \quad \varepsilon_k(k_t) = \frac{k_t f_k(k_t)}{f(k_t)} \in (0,1) \tag{7}$$

(cont.)

The law of motion of the augmented Solow model is

$$\frac{\dot{k}_t}{k_t} = \frac{sf(k_t)}{k_t} - (g + \delta + n)$$

whose log-linearization with respect to $\log k_t$ around $\log k^*$ gives

$$\frac{\dot{k}_t}{k_t} \simeq -(g + \delta + n)(1 - \varepsilon_k(k^*))(\log k_t - \log k^*) \quad (8)$$

Substituting (8) in (7) and $\log k_t - \log k^* = \frac{\log y_t - \log y_t^*}{\varepsilon_k(k^*)}$ obtained from the log-linearization of $\log y_t$ we have

$$\frac{\dot{y}_t}{y_t} = g - (g + \delta + n)(1 - \varepsilon_k(k^*))\left(\log y_t - \log y_t^*\right)$$

which finally relates output growth rate with the distance from the steady state. As we expected, the more distant, the faster convergence.

Growth regression 1

We can empirically test the implication that countries far-away from the steady state grow faster, by means of a simple regression:
In discrete time

$$g_{it+1,t} = b^0 + b^1 \log y_t + \varepsilon_i$$

where b^0 is the constant which captures the steady state and b^1 is the convergence parameter. Our test is that $b^1 < 0$. Precisely this is the test of *unconditional convergence* because we have assumed that all countries have the same steady state.

Unfortunately, unconditional convergence does not hold.

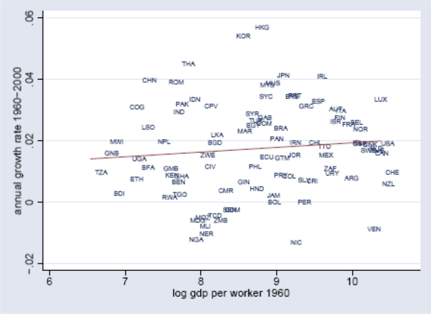


FIGURE 1.13. Annual growth rate of GDP per worker between 1960 and 2000 versus log GDP per worker in 1960 for the entire world.

Indeed it is too demanding: different countries might have different technologies, resources, institutions... so they might have different steady states: after all Solow predicts that each economy converges to its own steady state (*conditional convergence*).

Growth regression 3

Let's allow for different steady states and estimate:

$$g_{it,t-1} = b_i^0 + b^1 \log y_{it-1} + \varepsilon_{it}$$

with b_i^0 country specific.
Let's also model $b_i^0 = X_{it}\beta$ where X_{it} are country characteristics, such as

- male and female schooling rate
- fertility rate
- investment rate
- openness
- institutions

Growth regression 4

Barro (1991), Barro & Sala-i-Martin (1992, 2004) have estimated

$$g_{it,t-1} = X_{it}\beta + b^1 \log y_{it-1} + \varepsilon_{it}$$

and have found support for the hypothesis $b^1 < 0$.
They also claim that X_{it} are the ultimate causes of economic growth

THE AFRICA DUMMY IN FOUR GROWTH REGRESSIONS

Variable	Barro-Lee	Easterly-Levine	Sachs-Warner	Collier-Gunning
Period covered	1965–85	1960–89	1965–90	1960–89
Policies				
Investment/GDP	0.078(0.028)			0.0013[4.72]
Africa*Inv/GDP				0.00043[2.48]
Openness	0.033(0.0087)	0.020[4.63]	8.48[3.41]	0.0184[5.21]
Log GDP*Openness			–0.77[2.52]	
Africa*Openness				0.0111[1.88]
Institutions			0.28[3.49]	
Financial depth		0.015[2.54]		
Fiscal stance	–0.131(0.037)	–0.088[2.88]	0.12[5.34]	
Initial Conditions				
Initial Income	–0.026(0.0038)	0.066[2.69]	–1.63[7.89]	
Idem squared		–0.005[3.10]		–0.0009[6.97]
Labor force-pop.			1.20[3.41]	
Landlocked			–0.58[2.63]	–0.00789[2.40]
ELF		–0.016[2.54]		–0.0148[3.76]
Male schooling	0.0090(0.0044)			
Female schooling	–0.0052(0.0047)			
Schooling		0.009[2.28]		0.0148[1.64]
Life expectancy	0.0712(0.0148)		45.53[2.58]	

Empirical problems with growth regressions 1

- X_{it} and y_{it-1} are endogenous: they can be jointly determined with $g_{it,t-1}$ because
  - regarding y_{it-1}, the factors which make a country poor at t-1 might also penalize subsequent growth
  - regarding X_{it} the factors which cause a country to invest little in, say, human capital might also penalize growth
  - measurement error in variables
- Including the investment rate is at odds with the logic of the Solow model, where investment is the engine of growth
- Solow model assumes a closed economy! The growth equation embodies this assumption.

Empirical problems with growth regressions 2

A possible way to improve the growth regression is that of including country and period fixed effects

$$g_{it,t-1} = X_{it}\beta + b^1 \log y_{it-1} + \delta_i + \mu_t + \varepsilon_{it}$$

Fixed effects would capture any constant omitted variable. But:

- there might be time-varying omitted variables which are still not controlled for.
- if X_{it} change slowly, including fixed effects would make difficult to identify the effect of X_{it} and would emphasize measurement errors.

The Ramsey model

The paradigm of neo-classical growth models

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Introduction

- In the Solow model, the saving rate was exogenous
- In the Ramsey model (and in all other neoclassical models) consumers preferences are introduced and saving is endogenously generated into the model (but it fully depends on preferences which are exogenously assumed once again)

The Ramsey model in pills

- preferences generate savings
- capital accumulation generates short-run growth
- at the steady state capital and output per capita are constant
- the saving rates is always optimal and the economy at the steady state is at the "modified" golden rule
- optimal control model

The Ramsey model - hypothesis

- infinite horizon/continuos time
- representative household with instantaneous utility function $u(c_t)$ and infinite life,
  with $u'(c_t) > 0$, $u''(c_t) < 0$ and
  $\lim_{c \to 0} u'(c) = +\infty$, $\lim_{c \to \infty} u'(c) = 0$
- household population grows at rate n starting from $L_0 = 1$, so that $L_t = \exp(nt)$
- each individual inelastically supplies one unit of labor
- representative firm with "aggregate" technology $Y_t = F(K_t, L_t)$ with constant returns to scale and Inada conditions satisfied
- capital depreciation at rate δ

Firms

Technology in per capita terms:

$$\frac{Y_t}{L_t} = y_t = \frac{1}{L_t}F(K_t, L_t) = F\left(\frac{K_t}{L_t}, 1\right) = f(k_t) \text{ with } k_t = \frac{K_t}{L_t}$$

Firms behave competitively, they are price takers and maximize profits. The corresponding FOCs are

$$F_K(K_t, L_t) = R_t = f'(k_t) \quad (1)$$

$$F_L(K_t, L_t) = w_t = f(k_t) - k_t f'(k_t) \quad (2)$$

Proof

To prove (1) recall the Euler theorem: if $F(.)$ is homogenous of degree 1, $F_K(.)$ and $F_L(.)$ are homogenous of degree zero

$$F_K(K_t, L_t) = F_K\left(\frac{K_t}{L_t}, 1\right) = f'(k_t)$$

To prove (2), the Euler theorem states that $F(K_t, L_t) = F_K(.)K_t + F_L(.)L_t$ which implies

$$\frac{F(K_t, L_t)}{L_t} = F_K(.)\frac{K_t}{L_t} + F_L(.)$$

or

$$F_L(.) = f(k_t) - k_t f'(k_t)$$

Households

Individuals live forever and must maximize intertemporal utility by deciding the stream of instantaneous consumption. Given the flow of income, households can either save or borrow in order to meet the optimal consumption path. This amounts to solve a continuous-time maximization problem.

Households financial assets A_t, i.e. savings, evolve as

$$\dot{A}_t = r_t A_t + w_t L_t - c_t L_t$$

where

- r_t is the net rate of return of financial assets
- $r_t A_t$ is the flow of capital returns
- $w_t L_t$ is the flow of household labor income
- $c_t L_t$ is the flow of household consumption and c_t is per-capita consumption

In per-capita terms

$$\frac{\dot{A}_t}{L_t} = r_t \frac{A_t}{L_t} + w_t - c_t$$

Let be

$$a_t = \frac{A_t}{L_t}$$

Differentiate a_t with respect to time

$$\dot{a}_t = \frac{\dot{A}_t L_t - \dot{L}_t A_t}{L_t^2} = \frac{\dot{A}_t}{L_t} - n a_t$$

So that

$$\dot{a}_t = (r_t - n)\, a_t + w_t - c_t$$

Capital Market

Each household is free to borrow from the capital market at the rate of interest r_t. In order to avoid a Ponzi-game, i.e. the situation where past debts and interests are paid with new debts, we need to impose the condition

$$\lim_{t \to \infty} a_t \exp\left(-\int_0^t [r(s) - n]\, ds\right) \geqslant 0$$

This is the Net Present Value (NPV) of assets. This condition requires that in a sufficiently large neighborhood of $+\infty$ debts grow at a rate lower than $r(s) - n$.

Competitive equilibrium

Definition

A competitive equilibrium of the Ramsey economy is the sequence of paths

$$[c_t, k_t, w_t, R_t]_{t=0}^{\infty}$$

such that

- households maximize intertemporal utility subject to the law of accumulation of assets and the no-Ponzi-Game condition
- factor prices clear the factor markets
- interest rate clears the capital market

The intertemporal maximization

Instantaneous household utility is $u(c_t)L_t = u(c_t)\exp(nt)$. Household maximize their intertemporal discounted utility. The discount rate is ρ subject to the law of motion of assets and the no-Ponzi-game condition

$$\max_{c_t} \int_0^\infty u(c_t)\exp(-(\rho - n)t)dt$$

$$\text{s.t. } \dot{a}_t = (r_t - n)\, a_t + w_t - c_t$$

$$\lim_{t\to\infty} a_t \exp\left(-\int_0^t [r(s) - n]\, ds\right) \geqslant 0$$

This is an infinite horizon *optimal control problem.*

Optimal control problem

To solve this problem we use standard techniques of optimal control. Precisely, we write the associated Hamiltonian (in current value), neglecting the no-Ponzi-game condition that we will check ex-post.

$$H(a_t, c_t, \mu_t) = u(c_t) + \mu_t \left[(r_t - n) a_t + w_t - c_t \right]$$

where a_t is the state variable, c_t is the control variable and μ_t is the co-state variable. From the Hamiltonian we derive the following necessary conditions:

$$\frac{\partial H}{\partial c_t} = 0$$

$$\frac{\partial H}{\partial a_t} = -\dot{\mu}_t + (\rho - n)\mu_t$$

$$\lim_{t \to \infty} a_t \mu_t \exp\left(-\int_0^t (\rho - n) ds \right) = 0$$

where the last condition is the transversality (or boundary) condition.

Optimal solution (1)

$$\frac{\partial H}{\partial c_t} = u'(c_t) - \mu_t = 0 \tag{3}$$

$$\frac{\partial H}{\partial a_t} = (r_t - n)\,\mu_t = -\dot{\mu}_t + (\rho - n)\mu_t \tag{4}$$

From (4) we get

$$\frac{\dot{\mu}_t}{\mu_t} = \rho - r_t$$

By differentiating (3) with respect to time we get

$$u''(c_t)\dot{c}_t = \dot{\mu}_t$$

and dividing by (3) we have

$$\frac{u''(c_t)\dot{c}_t}{u'(c_t)} = \frac{\dot{\mu}_t}{\mu_t} \Longleftrightarrow -\sigma_u(c_t)\frac{\dot{c}_t}{c_t} = \rho - r_t$$

Optimal solution (cont.)

where

$$\sigma_u(c_t) = -\frac{u''(c_t)c_t}{u'(c_t)} > 0$$

is the elasticity of $u'(c_t)$, which measures the willingness of households to substitute consumption over time. Finally we can write

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma_u(c_t)}(r_t - \rho)$$

which represents the instantaneous growth rate of c_t. Therefore by integration we get the equilibrium path of c_t, which is

$$c_t = c_0 \exp\left(\int_0^t \frac{r_s - \rho}{\sigma_u(c_s)} ds\right) \tag{5}$$

Optimal solution (cont.)

Integrating $\frac{\dot{\mu}_t}{\mu_t} = \rho - r_t$, the instantaneous growth rate of μ_t we get, thanks to (3)

$$\mu_t = \mu_0 \exp\left(-\int_0^t (r_s - \rho)ds\right) = u'(c_0)\exp\left(-\int_0^t (r_s - \rho)ds\right) \qquad (6)$$

By substituting (6) into the transversality condition we get

$$\lim_{t\to\infty} u'(c_0)\left[a_t \exp\left(-\int_0^t (r_s - n)ds\right)\right] = 0$$

which implies the no-Ponzi-game condition. Therefore the optimal solution of the intertemporal maximization problem spontaneously meets the no-Ponzi-game condition.

Equilibrium

At equilibrium of this closed economy it must be that (per-capita) financial assets a_t coincide with (per capita) physical capital, the only available type of investment. This means also that $\dot{a}_t = \dot{k}_t$ and that the return to assets r_t must be equal to the net return to physical capital (= rental price - depreciation), i.e.

$$r_t = R_t - \delta = f'(k_t) - \delta$$

By knowing the path of c_t, the path of k_t is implicitly determined and the competitive equilibrium is of this economy is then

$$c_t = c_0 \exp\left(\int_0^t \frac{r_s - \rho}{\sigma_u(c_s)} ds\right)$$

$$\begin{aligned}
\dot{k}_t &= (r_t - n)\, k_t + w_t - c_t \\
&= \left[f'(k_t) - \delta - n\right] k_t + \left[f(k_t) - k_t f'(k_t)\right] - c_0 \exp\left(\int_0^t \frac{f'(k_s) - \delta - \rho}{\sigma_u(c_s)}\right.
\end{aligned}$$

Pareto Optimality

Note:

- households live forever and maximize consumption at the equilibrium. Moreover at equilibrium all markets clear. We are in a situation of "dynamic" general equilibrium. Therefore the equilibrium outcome is Pareto-optimal.
- given that the economy is at a Pareto-point, then there is no way to improve utility/consumption of an individual without harming someone else.
- intertemporal utility is maximized by construction (not as in Solow) and there is Pareto-efficiency (not as in the OLG model). Therefore the economy operates in a situation of "golden rule".

Steady State

The Ramsey model is characterized by a system of two dynamic equations $(\dot{c}_t, \dot{k}_t)$. Therefore the corresponding steady state must satisfy

$$\begin{aligned}\dot{c}_t &= 0\\ \dot{k}_t &= 0\end{aligned}$$

Given the dynamics of $\dot{c}_t$, at the steady state,

$$\dot{c}_t = 0 \text{ implies } r_t - \rho = 0 \text{ i.e.}$$

$$f'(k^*) = \rho + \delta \tag{7}$$

As at the equilibrium we maximize intertemporal utility (rather than steady state consumption), equation (7) is known as modified golden rule and defines the level of capital per-capita of steady state which maximizes intertemporal utility.

Given the dynamics of $\dot{k}_t$, the corresponding steady state consumption is

$$\begin{aligned}c^* &= r^*k^* + w^* = \left[f'(k^*) - \delta - n\right] k^* + \left[f(k^*) - k^*f'(k^*)\right] =\\ &= f(k^*) - (\delta + n)k^*\end{aligned}$$

Steady State (cont.)

Note:

- the level k_{gold} which maximizes consumption at the steady state would be

$$f'(k_{gold}) = \delta + n$$

  i.e. the steady state corresponding to the Solow model.

- In the Ramsey model individuals are impatient and prefer earlier consumption to later consumption. They choose the consumption path which is optimal according to their preferences and generate the modified golden rule

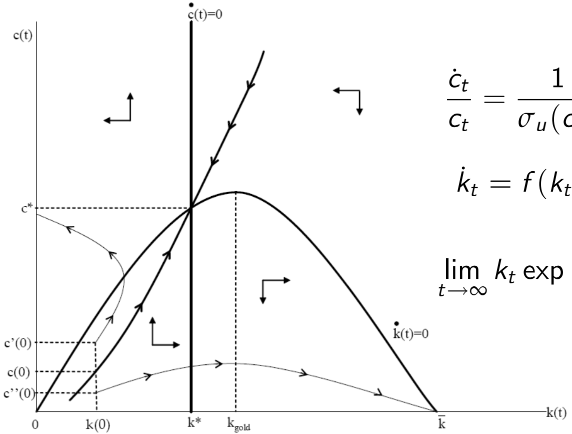
$$f'(k^*) = \rho + \delta$$

- Certainly $\rho > n$, so that $k^* < k_{gold}$. Therefore in Ramsey we will never be in a situation of dynamic inefficiency.

Steady State (cont.)

- As in the Solow model there is no sustained growth at the steady state, as capital and income per capita remain constant.
- This effect is entirely due to the diminishing marginal returns to capital: *intuitively, after a certain level of capital per-capita, obtaining one additional unit of future output (and consumption) requires to install a lot of additional capital at present, which must be financed by reducing a lot current consumption.*

Transitional Dynamics



$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma_u(c_t)} \left(f'(k_t) - \delta - \rho \right)$$

$$\dot{k}_t = f(k_t) - (\delta + n) k_t - c_t$$

$$\lim_{t \to \infty} k_t \exp\left(-\int_0^t (r_s - n) ds \right) = 0$$

Transitional Dynamics (cont.)

key results:

- there exists a unique stable arm, the path which passes through the steady state
- all other paths are not equilibrium path: all points out of the stable arm diverge and reach either zero consumption or zero capital stock
- paths which end up with $k_t = 0$ implies no production (there is no capital per worker) and there cannot be consumption (i.e. $c_t = 0$): this implies a jump towards the origin which is incompatible with the equation of the consumption equilibrium path.
- paths which end up with $c_t = 0$ violates the transversality condition which requires that the NPV of assets is zero.
- given the initial condition k_0, the essence of the optimization problem is that of adjusting c_0, i.e. deciding the initial level of consumption, in order to meet the unique equilibrium path.
- the equilibrium path is such that if $k_0 < k^*$ then both k_t and c_t increase and viceversa for $k_0 > k^*$

Formal characterization of the transition.

We can formally prove that there is a unique stable path (so called saddle path stability of the dynamic system) by linearizing the system

$$\dot{k}_t = f(k_t) - (\delta + n)k_t - c_t$$

$$\dot{c}_t = \frac{1}{\sigma_u(c_t)} \left(f'(k_t) - \delta - \rho\right) c_t$$

around (k^*, c^*) (i.e. approximation to a linear system).
We get:

$$\begin{aligned} \dot{k}_t &= \text{constant } + \left[f'(k^*) - (\delta + n)\right] (k_t - k^*) - (c_t - c^*) \\ \dot{c}_t &= \text{constant} + \frac{c^*}{\sigma_u(c^*)} f''(k^*)(k_t - k^*) + 0(c_t - c^*) \end{aligned}$$

Note: the last coefficient is 0 because $f'(k^*) = \delta + \rho$ for any c_t. This implies also that $f'(k^*) - (\delta + n) = \rho - n$

Formal characterization of the transition (cont.)

Now consider the matrix

$$A = \begin{bmatrix} \rho - n & -1 \\ \frac{c^*}{\sigma_u(c^*)} f''(k^*) & 0 \end{bmatrix}$$

of the coefficients of the linearized system.
A standard theorem on the stability of linear systems states that:

Theorem

Given a two-equation linear dynamic system, if the matrix of the coefficients A has one eigenvalue with negative real part and one eigenvalue with positive real part, then the system has a unique solution path which passes through the initial condition and converges to the steady state.

Formal characterization of the transition (cont.)

To find the eigenvalues of a matrix we need to compute the roots of the corresponding characteristic polynomial $P(A)$. The latter is defined as

$$P(A) = \det(A - \lambda I) = \det \begin{bmatrix} \rho - n - \lambda & -1 \\ \frac{c^*}{\sigma_u(c^*)} f''(k^*) & -\lambda \end{bmatrix}$$

i.e.

$$P(A) = \lambda^2 - \lambda(\rho - n) + \frac{c^*}{\sigma_u(c^*)} f''(k^*)$$

Finally $P(A) = 0$ if

$$\lambda_{1,2} = \frac{(\rho - n) \pm \sqrt{(\rho - n)^2 - 4\frac{c^*}{\sigma_u(c^*)} f''(k^*)}}{2}$$

one root is positive and one is negative given that $\frac{c^*}{\sigma_u(c^*)} f''(k^*) < 0$. Therefore the linearized system is saddle-path stable.

Technological progress.

- By adding technological progress, as we did into the Solow model, all the properties of the Ramsey model are maintained.
- There will exist a unique stable path which leads to a unique steady state, where income per effective unit of labor is stable, but where per-capita capital and income are increasing at the rate g, i.e. the rate of technological progress.
- Sustained growth is then completely determined by the technological progress.

The Lucas model (1988)

A model of endogenous growth

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Introduction

- In this model endogenous long-run growth is obtained by means of human capital production and accumulation
- Human capital is created endogenously by the economic system
- Human capital balances diminishing returns to physical capital
- Therefore there is (balanced) growth in the long run

On human capital

- Human capital can be thought as an other form of capital which is embodied in the workers.
- Then workers can save either in human capital (by going to school for instance) or in physical capital.
- Both forms of capital produce income.
- Of course the rate of return of both must be equal at equilibrium, otherwise everyone will save on the more profitable form of capital.

Two sectors

There are two productive sectors in this economy:

- the first produces the final good which is consumed
- the second re-produces human capital

In both sectors human capital is an input. Physical capital is employed only in sector 1.

Sector 1:

$$Y_t = F(K_t, H_{1t})$$

with constant returns to scale, where human capital employed in sector 1 is $H_{1t} = h_{1t} L_{1t}$

- h_{1t} is the amount of human capital per worker
- L_{1t} is the amount of labor

In per-capita terms

We transform the production function in per-capita terms

$$y_t = \frac{F(K_t, h_{1t}L_{1t})}{L_{1t}} = f(k_t, h_{1t})$$

and we assume

$$f(k_t, h_{1t}) = Ak_t^{\alpha}h_{1t}^{1-\alpha}$$

Human capital

Sector 2:
New human capital is produced by means of already produced human capital alone (e.g. teachers and students)

$$\dot{h}_t = \phi h_{2t}$$

As human capital is fully employed it must be that

$$h_t = h_{1t} + h_{2t}$$

The key problem is that of allocating human capital optimally between the two sectors

Firms

Sector 1
Firms behave competitively and maximize profits:

$$\max \pi_1 = Ak_t^{\alpha} h_{1t}^{1-\alpha} - w_t h_{1t} - r_t k_t$$

where w_t and r_t are rental prices of resp. human and physical capital. The price of the final good is normalized to 1 (numeraire)
FOCs are

$$\alpha A k_t^{\alpha-1} h_{1t}^{1-\alpha} = r_t \quad (1)$$

$$(1-\alpha) A k_t^{\alpha} h_{1t}^{-\alpha} = w_t \quad (2)$$

Firms (cont.)

Sector 2

Also firms in sector 2 maximize profits

$$\max \pi_2 = p_t \dot{h}_t - w_t h_{2t} = p_t \phi h_{2t} - w_t h_{2t}$$

where

- p_t is the relative price of human capital (in terms of the price of the final good)
- w_t is the rental price of human capital employed in sector 2. This price must be equal in the two sectors to make the market of human capital in equilibrium.

The FOC is

$$p_t \phi = w_t \tag{3}$$

Arbitrage condition

At equilibrium the rate of return of human and physical capital must be the same

$$\frac{w_t}{p_t} + \frac{\dot{p}_t}{p_t} = r_t \tag{4}$$

where

- $\frac{w_t}{p_t}$ is the return of human capital at time t
- $\frac{\dot{p}_t}{p_t}$ is the potential "capital gain" of human capital
- r_t is the return of physical capital

The path of p

The first equilibrium path we derive is that of the price of human capital: By substituting (1) and (3) in (4) we get

$$\frac{\dot{p}_t}{p_t} = \alpha A k_t^{\alpha-1} h_{1t}^{1-\alpha} - \phi = \alpha A \left(\frac{k_t}{h_{1t}}\right)^{\alpha-1} - \phi \tag{5}$$

Moreover by substituting (3) into (2) we get

$$(1-\alpha) A k_t^{\alpha} h_{1t}^{-\alpha} = p_t \phi$$

which can be written as

$$\frac{k_t}{h_{1t}} = \left(\frac{p_t \phi}{(1-\alpha)A}\right)^{\frac{1}{\alpha}} = (Qp)^{\frac{1}{\alpha}} \tag{6}$$

where

$$Q \equiv \frac{\phi}{(1-\alpha)A}$$

The path of p (cont.)

Finally by substituting (6) in (5) we obtain the equilibrium path of p_t

$$\frac{\dot{p}_t}{p_t} = \alpha A \left(Qp\right)^{\frac{\alpha-1}{\alpha}} - \phi$$

which is an equation on p_t alone

Consumers

There is a constant population of identical consumers which maximize

$$\max \int_0^\infty \exp(-\theta t) \frac{c_t^{1-\eta} - 1}{1-\eta} dt$$

under the intertemporal budget constraint

$$\dot{W}_t = r_t W_t - c_t \tag{7}$$

where

$$W_t = k_t + p_t h_t$$

is the value of assets hold by an individual at time t (i.e. wealth).

- Note: the rate of return of wealth is r_t thanks to the arbitrage condition

Optimal consumption path

The Hamiltonian corresponding to this problem is

$$H = \frac{c_t^{1-\eta} - 1}{1-\eta} + \mu_t \left(r_t W_t - c_t \right)$$

and the necessary conditions are

$$\frac{\partial H}{\partial c_t} = c_t^{-\eta} - \mu_t = 0 \tag{8}$$

$$\frac{\partial H}{\partial W_t} = \mu_t r_t = -\dot{\mu}_t + \theta \mu_t \tag{9}$$

$$\lim_{t \to \infty} \exp(-\theta t) \mu_t W_t = 0$$

Optimal consumption path (cont.)

From (9) we get

$$\frac{\dot{\mu}_t}{\mu_t} = \theta - r_t$$

By differentiating (8) wrt time and by dividing by (8) we get

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\eta}\left(r_t - \theta\right)$$

Equilibrium path of physical capital

1. By differentiating $W_t = k_t + p_t h_t$ wrt time we have
$$\dot{W}_t = \dot{k}_t + \dot{p}_t h_t + p_t \dot{h}_t$$
2. By equating to $r_t W_t - c_t$ (by using the budget constraint (7))
$$\dot{k}_t + \dot{p}_t h_t + p_t \dot{h}_t = r_t (k_t + p_t h_t) - c_t$$
3. By using the arbitrage condition (4) $w_t + \dot{p}_t = r_t p_t$ we get
$$\begin{aligned} \dot{k}_t &= r_t k_t + (w_t + \dot{p}_t) h_t - \dot{p}_t h_t - p_t \dot{h}_t - c_t \\ &= r_t k_t + w_t h_t - p_t \dot{h}_t - c_t \end{aligned}$$
4. Finally, by using $\dot{h}_t = \phi h_{2t}$ and $p_t \phi = w_t$ we get
$$\begin{aligned} \dot{k}_t &= r_t k_t + w_t (h_{1t} + h_{2t}) - p_t \dot{h}_t - c_t \\ &= r_t k_t + w_t h_{1t} - c_t \\ &= A k_t^{\alpha} h_{1t}^{1-\alpha} - c_t \end{aligned}$$

since $r_t k_t + w_t h_{1t} = A k_t^{\alpha} h_{1t}^{1-\alpha}$ thanks to constant returns to scale

The equilibrium dynamic system

Putting pieces together, by using $\frac{k_t}{h_{1t}} = (Qp)^{\frac{1}{\alpha}}$ (equation (6)) the resulting equilibrium path are such that

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\eta}(r_t - \theta) = \frac{1}{\eta}\left(\alpha A k_t^{\alpha-1} h_{1t}^{1-\alpha} - \theta\right) = \tag{10}$$

$$= \frac{1}{\eta}\left(\alpha A \frac{k_t^{\alpha-1}}{h_{1t}^{\alpha-1}} - \theta\right) = \frac{1}{\eta}\left(\alpha A (Qp_t)^{\frac{\alpha-1}{\alpha}} - \theta\right)$$

$$\frac{\dot{k}_t}{k_t} = A(Qp_t)^{\frac{\alpha-1}{\alpha}} - \frac{c_t}{k_t} \tag{11}$$

$$\frac{\dot{h}_t}{h_t} = \phi \frac{h_{2t}}{h_t} = \phi \frac{h_t - h_{1t}}{h_t} = \phi \left(1 - (Qp)^{\frac{1}{\alpha}} \frac{k_t}{h_t}\right) \tag{12}$$

$$\frac{\dot{p}_t}{p_t} = \alpha A (Qp)^{\frac{\alpha-1}{\alpha}} - \phi \tag{13}$$

Balanced growth.

Before solving the four dimension system, we need to check whether it can generate (it is compatible with) balanced growth path.

- Define γ_c and γ_k the long run *constant* growth rate of consumption and capital.
- In the long run it must be that $\dot{p}_t = 0$, i.e. the relative price of human capital relative to the final good must be constant at equilibrium (otherwise inflation bubble...). Therefore by setting $\alpha A \left(Qp \right)^{\frac{\alpha-1}{\alpha}} - \phi = 0$ we have

$$\overline{p} = \frac{1}{Q} \left[\frac{\phi}{\alpha A} \right]^{\frac{\alpha}{\alpha-1}} \tag{14}$$

- Substituting in the dynamics of c_t (10) we have

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\eta} \left[\phi - \theta \right] = \gamma_c \tag{15}$$

which is certainly constant

Balanced growth (cont.)

- Now by substituting $\overline{p}$ in the dynamics of k_t we have

$$\frac{\dot{k}_t}{k_t} = \frac{\phi}{\alpha} - \frac{c_t}{k_t}$$

which can be rewritten as

$$\frac{c_t}{k_t} = \frac{1}{\alpha}\left(\eta\gamma_c + \theta\right) - \gamma_k$$

by substituting equation (15). By assuming γ_k constant, the right-hand side is constant. Therefore it must be that consumption and physical capital grow at the same rate, i.e.

$$\gamma_c = \gamma_k$$

- Finally, by looking at the dynamics of h_t, evaluated at $\overline{p}$, we realize that $\frac{k_t}{h_t}$ must be constant in order to have γ_h constant, so that

$$\gamma_k = \gamma_h$$

Balanced growth (cont.)

Summing up, the Lucas model can produce long run balanced growth with

$$\gamma_c = \gamma_k = \gamma_h = \frac{1}{\eta}\left[\phi - \theta\right]$$

Precisely, there will be positive growth in the long run if the productivity in the human capital sector ϕ is larger than θ.
The engine of long run growth is the human capital sector, whose continuous production of new human capital keeps "high" the marginal productivity of physical capital, at all levels of physical capital.

$$MP_k = \alpha A k_t^{\alpha-1} h_{1t}^{1-\alpha} = \alpha A \left(\frac{k_t}{h_t}\right)^{\alpha-1}$$

Actually, along the path of balanced growth, the marginal productivity of physical capital will remain constant!
This allows to escape from the trap of diminishing returns.

Transition Dynamics

The study of the transition dynamics serves to prove that the model will actually produce balanced growth at equilibrium, i.e. that balanced growth is not a mere possibility, but it actually realizes..
Let's reduce the dimensionality of the system by setting

$$x_t = \frac{c_t}{k_t} \quad \text{and} \quad z_t = \frac{k_t}{h_t}$$

Therefore

$$\frac{\dot{x}_t}{x_t} = \frac{\dot{c}_t}{c_t} - \frac{\dot{k}_t}{k_t} \quad \text{and} \quad \frac{\dot{z}_t}{z_t} = \frac{\dot{k}_t}{k_t} - \frac{\dot{h}_t}{h_t}$$

Transition Dynamics (cont.)

By substitution, the reduced dynamic system is

$$\frac{\dot{x}_t}{x_t} = \frac{1}{\eta}\left[\theta - (\alpha - \eta)A(Qp)^{\frac{\alpha-1}{\alpha}}\right] - x_t$$

$$\frac{\dot{z}_t}{z_t} = A(Qp)^{\frac{\alpha-1}{\alpha}} - x_t - \phi + \phi(Qp)^{-\frac{1}{\alpha}} z_t$$

$$\frac{\dot{p}_t}{p_t} = \alpha A\left(Qp\right)^{\frac{\alpha-1}{\alpha}} - \phi$$

Has this system a steady state? If so at the steady state we will have balanced growth in c_t, k_t, h_t.

Steady state

$$\begin{cases} \dot{x}_t = 0 \\ \dot{z}_t = 0 \\ \dot{p}_t = 0 \end{cases} \iff \begin{cases} \overline{x}_t = \frac{1}{\eta}\left[\theta - (\alpha - \eta)\frac{\phi}{\alpha}\right] \\ \overline{z}_t = \left(\frac{\alpha A}{\phi}\right)^{\frac{1}{1-\alpha}} \frac{\theta-(1-\eta)\phi}{\eta\phi} \\ \overline{p}_t = \frac{1}{Q}\left(\frac{\phi}{\alpha A}\right)^{\frac{\alpha}{\alpha-1}} \end{cases}$$

There exists a unique steady state as the system is linear in x_t, z_t.
At the steady state,

- the ratio of consumption and capital per capita is constant
- the ratio between physical and human capital is constant
- this implies that the three quantities must increase at the same rate, which is constant because consumption grows at a constant rate.

Convergence to the steady state?

The final point to check is whether the economy, when at equilibrium, is converging to the steady state.

- Actually, we shall show that there exists a unique equilibrium path and it will converge to the steady state (saddle-path stability).

Formal analysis of convergence.

To study the equilibrium dynamics of each (non-linear) dynamic system we need to linearize it around the steady state:

$$
\begin{aligned}
\dot{p}_t &= m(p_t) = m(\overline{p}_t) + m'_p(\overline{p}_t)(p_t - \overline{p}_t) \\
\dot{x}_t &= f(p_t, x_t) \simeq f(\overline{p}_t, \overline{x}_t) + f'_p(\overline{p}_t)(p_t - \overline{p}_t) + f'_x(\overline{x}_t)(x_t - \overline{x}_t) \\
\dot{z}_t &= g(p_t, x_t, z_t) \simeq g(\overline{p}_t, \overline{x}_t, \overline{z}_t) + g'_p(\overline{p}_t)(p_t - \overline{p}_t) + \\
&\quad + g'_x(\overline{x}_t)(x_t - \overline{x}_t) + g'_z(\overline{z}_t)(z_t - \overline{z}_t)
\end{aligned}
$$

Next we consider its Jacobian (the matrix of partial derivatives)

$$
J = \begin{bmatrix} m'_p(\overline{p}_t) & 0 & 0 \\ f'_p(\overline{p}_t) & f'_x(\overline{x}_t) & 0 \\ g'_p(\overline{p}_t) & g'_x(\overline{x}_t) & g'_z(\overline{z}_t) \end{bmatrix}
$$

which is a triangular matrix.

Theorem.

Theorem

Consider a linearized dynamic system

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = J \begin{bmatrix} X \\ Y \end{bmatrix}$$

where X is the set of predetermined variables (i.e. the variables with an initial condition) and Y the set of non-predetermined variables. If the number of unstable eigenvalues (i.e. those with positive real parts) of J is equal to the number of non-predetermined variables, then the dynamic systems has a unique solution. Under the same conditions, if all unstable eigenvalues are also real and distinct, then the solution is a saddle path.

Formal analysis of convergence (cont.)

- In our case we have that only z_t is predetermined with an initial condition $z_0 = \frac{k_0}{h_0}$.
- Instead x_t and p_t are not predetermined.
- The eigenvalues of our triangular matrix are the diagonal entries
- In our case, $m'_p(\overline{p}_t) > 0$, $f'_x(\overline{x}_t) < 0$ and $g'_z(\overline{z}_t) > 0$ and they are all distinct real numbers.
- Therefore saddle-path stability will emerge.

Final note

- It is possible to prove that in the Lucas model the endogenous growth rate obtained by a perfectly competitive economy is socially optimal, i.e. it corresponds to the path which a benevolent social planner would choose in order to maximize people welfare.
- In other words a competitive economy will achieve the highest possible level of welfare: this does not mean however the highest possible long run growth rate (which might require too much saving...).
- Here, as in Ramsey, the first welfare theorem holds because there are no market imperfections or market failures.